

Lovász Local Lemma and the resampling algorithm

3-SAT

"conjunctive normal form"

CNF formula

$$\varphi(x_1, \dots, x_n) = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_4 \vee \dots) \wedge \dots$$

m clauses, n variables

3 (distinct) variables per clause

The silliest APX algorithm

1. for $i = 1, \dots, n$

a. randomly set $x_i = \begin{cases} \text{TRUE} & \text{w/ prob .5} \\ \text{FALSE} & \text{w/ prob .5} \end{cases}$

We will:

① prove silly algo is $\frac{7}{8}$ -APX

② derandomize silly algo $\Rightarrow$ deterministic $\frac{7}{8}$ -APX

Even more surprising:

"probabilistically checkable proofs"

PCP Theorem

$(\frac{7}{8} + \epsilon)$ -APX is NP-Hard $\forall \epsilon > 0$

Approximating 3SAT: "Max 3-SAT"

$$\varphi(x_1, \dots, x_n) = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_4 \vee \dots) \wedge \dots$$

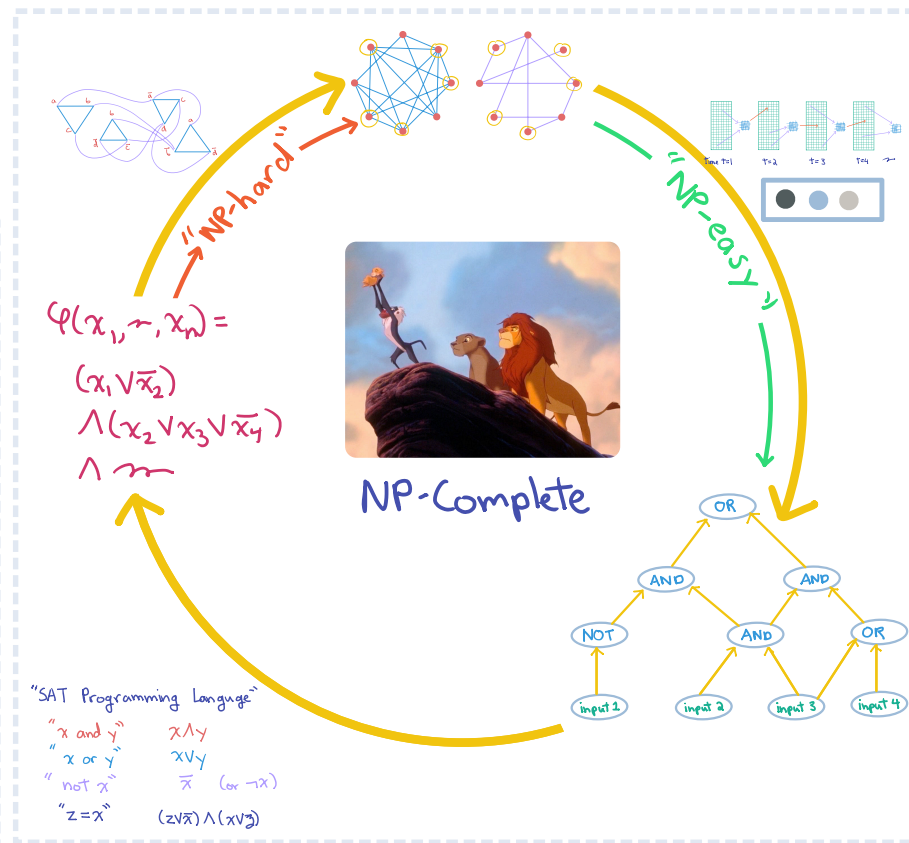
Goal: satisfy Max # clauses

Interesting even if φ is unsatisfiable

"d-Approximation algorithm" $d \in (0, 1)$

Algorithm satisfies $d \cdot \text{OPT}$ clauses

Optimum (max) value



Suppose $m < 2^k$
(# clauses)

for random assignment $x_1, \dots, x_n \in \{0,1\}^n$:

$$\Pr[\varphi(x_1, \dots, x_n) = \text{false}]$$

$$\leq \sum_{i=1}^m \Pr[\text{i-th clause unsatisfied}]$$

$$\leq m \cdot \frac{1}{2^k} < 1$$

$\Rightarrow \varphi$ must be satisfiable!

for $m > 2^k$, union bound not strong enough...

Resampling algorithm (for k-SAT)

1. Sample $x_1, \dots, x_n \in \{T, F\}$ (independently, uniformly at random)
2. while there is an unsatisfied clause C :
 - a. resample each x_i in C (independently, uniformly at random)
3. return $x_1, \dots, x_n$

Resampling algorithm (for k-SAT) (Moser, 2009)

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Theorem Let $\mathcal{f}(x_1, \dots, x_n)$ be a k-SAT formula where each clause intersects at most d other clauses.

If $d \leq \frac{2^k}{e} - 1$, then

- $\mathcal{f}$ is satisfiable.
- resampling algo finds satisfying assignment while resampling each clause $\frac{1}{d}$ times in expectation

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Theorem Let $\phi(x_1, \dots, x_n)$ be a k-SAT formula where each clause intersects at most d other clauses.

If $d \leq \frac{2^k}{e} - 1$, then

locally verifiable

• ϕ is satisfiable. $\leftarrow$ 1970's

• resampling algo finds satisfying assignment while resampling each clause $\frac{1}{d}$ times in expectation

2009

Lovász Local Lemma

(k-SAT is special case)

$A_1, \dots, A_n$ events

(e.g. A_i = i th clause unsatisfied)

" $A_i \sim A_j$ " $\Rightarrow$ A_i and A_j not mutually independent

(e.g. $A_i \sim A_j$ is i th and j th clause intersect)

Q: Is it possible for none of $A_1, \dots, A_n$ to occur?

Is $P[\bar{A}_1, \bar{A}_2, \dots, \bar{A}_n] > 0$?

YES if $A_1, \dots, A_n$ are independent

NO w/ no assumptions on dependence

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[Erdős and Lovász, 1975]

Theorem: Suppose $\exists \chi(A_1), \dots, \chi(A_n) \in (0,1)$ s.t.

$$P[A_i] \leq \chi(A_i) \prod_{\substack{A_j \sim A_i \\ A_j \neq A_i}} (1 - \chi(A_j)) \quad \forall A_i.$$

Then $P[\bar{A}_1, \dots, \bar{A}_n] > 0$

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Then $P[\bar{A}_1, \dots, \bar{A}_n] > 0$

Corollary: Suppose each A_i is independent from all but d other events A_j , and $P[A_i] \leq p \quad \forall A_i$.

If $e_p(d+1) \leq 1$, then $P[\bar{A}_1, \dots, \bar{A}_n] > 0$.

Proof: set $x(A_i) = \frac{1}{d+1} \quad \forall A_i$

Theorem: Suppose $\exists x(A_1), \dots, x(A_n) \in (0,1)$ s.t.
 $P[A_i] \leq x(A_i) \prod_{\substack{A_j \sim A_i \\ A_j \neq A_i}} (1 - x(A_j)) \quad \forall A_i.$

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Corollary: Suppose each A_i is independent from all but d other events A_j , and $P[A_i] \leq p \quad \forall A_i.$

If $ep(d+1) \leq 1$, then $P[\bar{A}_1, \dots, \bar{A}_n] > 0.$

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If $d \leq \frac{2^k}{e} - 1$, then

• ϕ is satisfiable.

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Lovász Local Lemma - variable setting (k-SAT is special case)

$A_1, \dots, A_n$ events

(e.g. A_i = i th clause unsatisfied)

" $A_i \sim A_j$ " $\Rightarrow A_i$ and A_j not mutually independent

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Variables $Y = \{X_1, X_2, \dots\}$

(e.g., $x_1, \dots, x_n \in \{T, F\}$)

- Each X_i independent w/ its own distribution
- Each A_i determined by subset $S_i \subseteq Y$
- " $A_i \sim A_j$ " $\Rightarrow S_i \cap S_j \neq \emptyset$

(Moser-Tardos, 2010)

Lovász Local Lemma - variable setting
(k-SAT is special case)

$A_1, \dots, A_n$ events (e.g. A_i = i th clause unsatisfied)

" $A_i \sim A_j$ " $\Rightarrow A_i$ and A_j
~~not~~ mutually independent
(e.g. $A_i \sim A_j$ is i th and j th clause intersect)

Variables $Y = \{X_1, X_2, \dots\}$ (e.g. $x_i, \neg x_i \in \{T, F\}$)

- Each X_i independent w/ its own distribution
- Each A_i determined by subset $S_i \subseteq Y$
- " $A_i \sim A_j$ " $\Rightarrow S_i \cap S_j \neq \emptyset$

Resampling algorithm (for general LLL)

1. Sample all variables Y (independently, from their respective distributions)
2. while there is an event A_i
 - a. resample all of S_i (independently, from their respective distributions)

Resampling algorithm (for general LL) (Moser-Tardos, 2010)

1. Sample all variables V (independently, from their respective distributions)
2. while there is an event A_i :
 - a. resample all of S_i (independently, from their respective distributions)

Theorem: Suppose $\exists x(A_1), \dots, x(A_n) \in (0,1)$ s.t.
$$P[A_i] \leq x(A_i) \prod_{\substack{A_j \sim A_i \\ A_j \neq A_i}} (1 - x(A_j)) \quad \forall A_i.$$

Then $P[\bar{A}_1, \dots, \bar{A}_n] > 0$, and the resampling algo finds a configuration of V inducing $\bar{A}_1, \dots, \bar{A}_n$ while resampling each A_i at most $\frac{x(A_i)}{1 - x(A_i)}$ times in expectation

Proof.

very subtle "coupling" argument

need to introduce:

1. "log"
2. "witness trees"

Resampling algorithm (for general LL) (Moser-Tardos, 2010)

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The "log"

Records resampled
events in chronological

- $A_1, A_7, A_{11}, A_{12}, \dots$

want to bound $E[|\log|]$

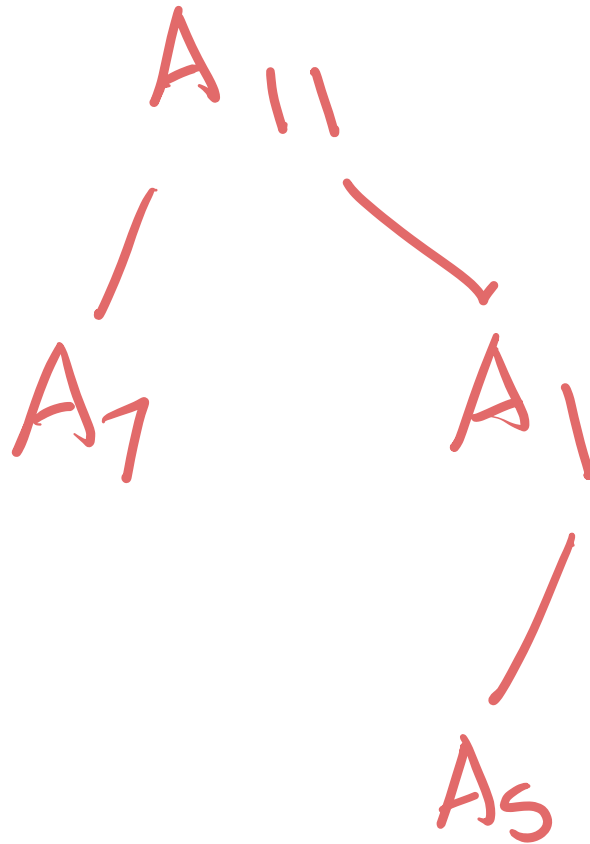
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Witness trees

Def: rooted tree w/ nodes labeled by events

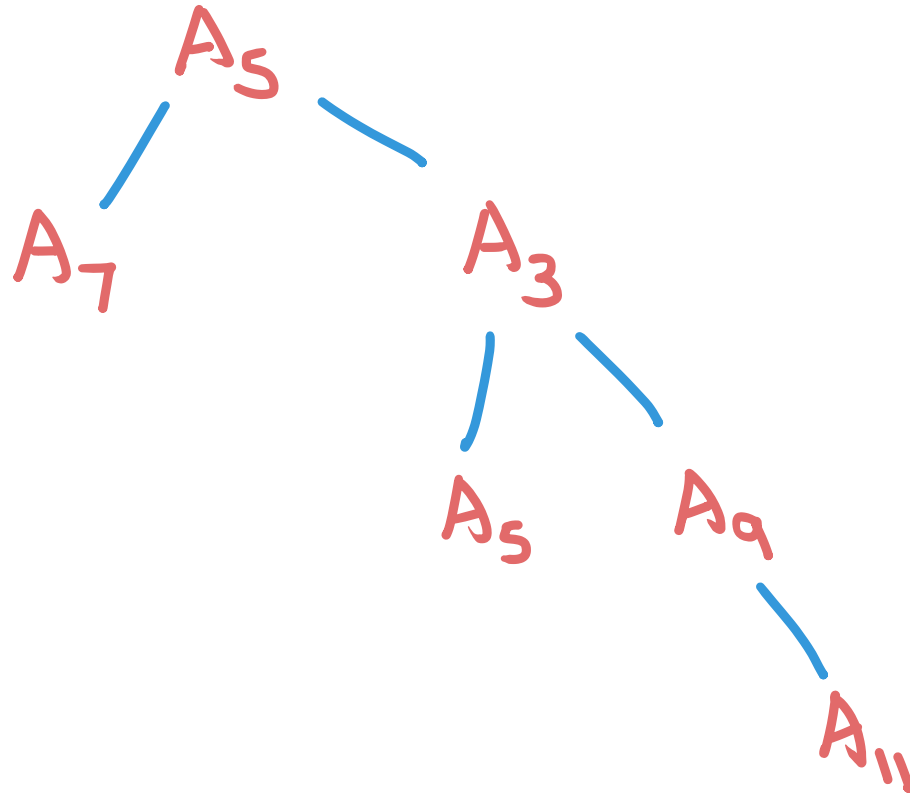
(labels can repeat)



Witness trees

Def: rooted tree w/ nodes labeled by events

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Mapping prefixes of the log $\rightarrow$ witness trees

Suppose first i events are $A_{i_1}, A_{i_2}, \dots, A_{i_i}$.

A_{i_i}

① Make root w/ label A_{i_i} .

② For $j = i-1, i-2, \dots, 1$

① If $A_{i_j} \sim A_k$ for some label A_k in the tree

① pick such a node N of max depth

② make a new child of N w/ label A_{i_j}

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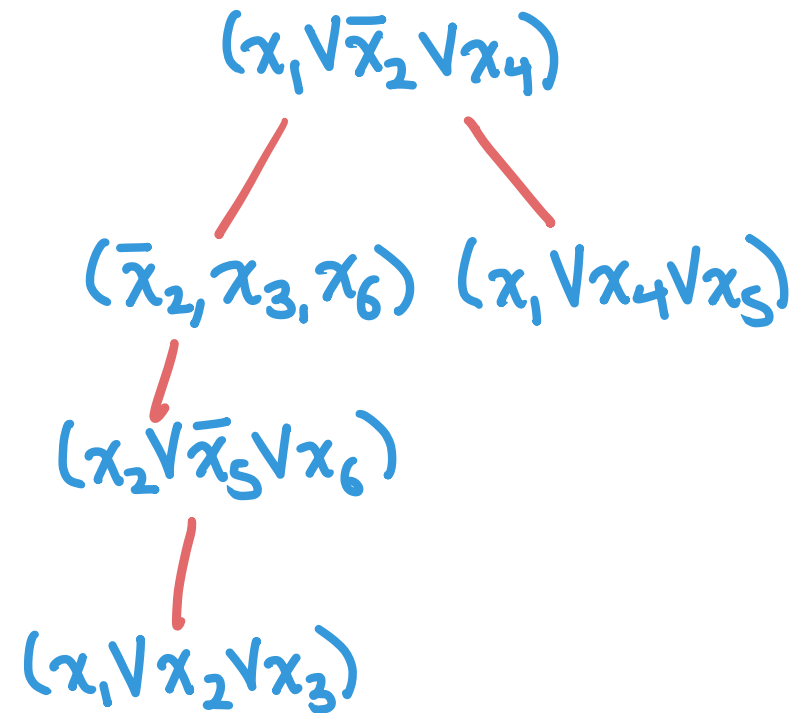
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Log

$(x_1 \vee x_2 \vee x_3), (x_2 \vee \bar{x}_5 \vee x_6), (x_1, \bar{x}_8, x_9),$
 $(x_1 \vee x_4 \vee x_5), (\bar{x}_2, x_3, x_6), (x_1 \vee \bar{x}_2 \vee x_4)$



Suppose first i events are $A_{i_1}, A_{i_2}, \dots, A_{i_i}$.

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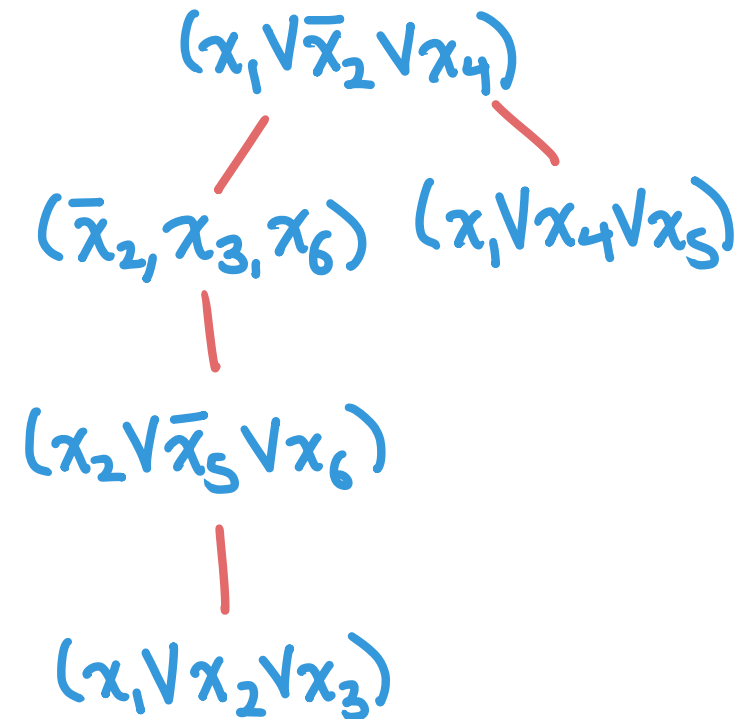
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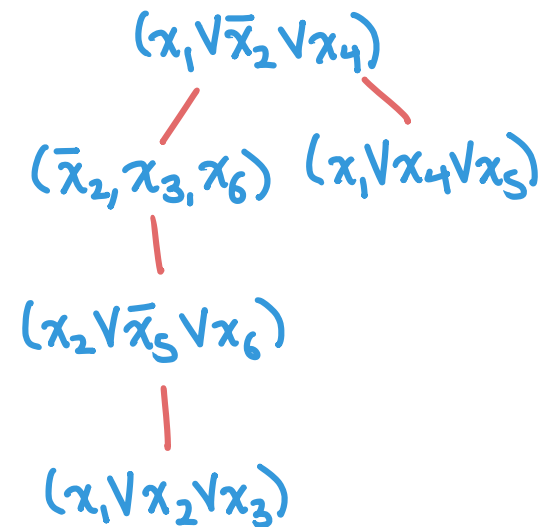
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Observation (for later)

Let T_ℓ = witness tree for $A_{i_1}, \dots, A_{i_\ell}$

Suppose $j < k$ and $A_{i_j} \sim A_{i_k}$

if A_{i_k} is a node in T_ℓ , then so is A_{i_j}

and $\text{depth}(A_{i_j}) > \text{depth}(A_{i_k})$



Fix A_i .

Goal: bound # times A_i resampled

let $W_i = \{\text{all witness trees w/ root } A_i\}$

for $j=1, 2, \dots$, let T_j be witness tree

for the first j entries of the log

(or $\emptyset$ if log has $< j$ entries)

$$E \left[\begin{array}{c} \# \text{ times } A_i \\ \text{resampled} \end{array} \right] = E \left[\begin{array}{c} \# \text{ times } A_i \\ \text{appears in log} \end{array} \right]$$
$$= \sum_{j=1}^{\infty} \sum_{T \in W_i} P_c [T_j = T]$$

Resampling algorithm (for general LL) (Moser-Tardos, 2010)

1. Sample all variables V (independently, from their respective distributions)
2. while there is an event A_i :
 - a. resample all of S_i (independently, from their respective distributions)

Theorem: Suppose $\exists \chi(A_i), \dots, \chi(A_n) \in (0,1)$ s.t.
 $P[A_i] \leq \chi(A_i) \prod_{\substack{A_j \sim A_i \\ A_j \neq A_i}} (1 - \chi(A_j)) \quad \forall A_i.$

Then $P[\bar{A}_1, \dots, \bar{A}_n] > 0$, and the resampling algo finds a configuration of V inducing $\bar{A}_1, \dots, \bar{A}_n$ while resampling each A_i at most $\frac{\chi(A_i)}{1 - \chi(A_i)}$ times in expectation

Mapping prefixes of the log $\rightarrow$ witness trees

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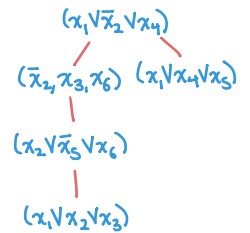
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(x_1, x_2, x_3), ($x_2, \bar{x}_5, x_6$), ($x_1, \bar{x}_8, x_9$),
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let $W_i = \{\text{all witness trees w/ root } A_i\}$

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(or $\emptyset$ if log has $< j$ entries)

$$E[\text{\# times } A_i \text{ resampled}] = E[\text{\# times } A_i \text{ appears in log}]$$

$$= \sum_{j=1}^{\infty} \sum_{T \in W_i} P_r[T_j = T]$$

$$= \sum_{T \in W_i} P_r[T_j = T \text{ for some } j]$$

Lemma let j, k index diss. entries in log
then $T_j \neq T_k$

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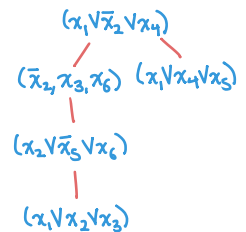
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Mapping prefixes of the log $\rightarrow$ witness trees

Suppose first i events are $A_{i1}, A_{i2}, \dots, A_{ii}$.

- ① Make root w/ label A_{ii} .
- ② For $j=i-1, i-2, \dots, 1$
 - Ⓐ If $A_{ij} \sim A_k$ for some label A_k in the tree
 - ① pick such a node N of max depth
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 $(x_1, x_2, x_3), (x_2, \bar{x}_5, x_6), (x_1, \bar{x}_8, x_9),$
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$$\Pr[T_j = T \text{ for some } j]?$$

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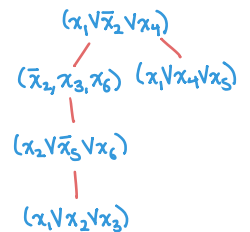
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let $p(T) = \prod P[A_j]$
labels A_j in T
(w/ multiplicity)

Lemma $\Pr[T_j = T \text{ for some } j] \leq p(T)$

Proof introduce "checking" procedure

check(T)

① For each event A_{ij} in T in decr. depth:
(label) (leaf first, root last)

① Sample variables in S_{ij} . If the sample does not induce A_{ij} , then return FAIL

② return SUCCESS

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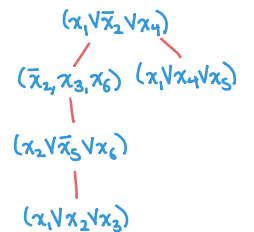
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Lemma $\Pr[T_j = T \text{ for some } j] \leq p(T) = \prod P[A_j]$
labels A_j in T
(w/ multiplicity)

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(label) (leaf first, root last)
- ⓐ Sample variables in S_{ij} . If the sample does not induce A_{ij} , then return FAIL
- ② return SUCCESS

$$\Pr[\text{check}(T) \text{ succeeds}] = p(T) = \prod P[A_j]$$

labels A_j in T
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claim: $\Pr[T_j = T \text{ for some } j] \leq \Pr[\text{check}(T) \text{ succeeds}]$

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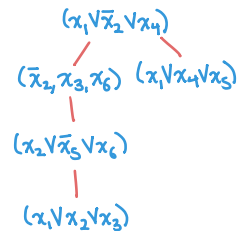
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Mapping prefixes of the log $\rightarrow$ witness trees

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Log
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claim: $\Pr[T_j = T \text{ for some } j] \leq \Pr[\text{check}(T)]$

Imagine generating all (repeated) samples for all variables ahead of time.

V	1st sample	2nd sample	3rd sample	4th sample	5th	6th	7th	
X_1	$X_1^{(1)}$	$X_1^{(2)}$	$X_1^{(3)}$	$X_1^{(4)}$	$X_1^{(5)}$	$X_1^{(6)}$	$X_1^{(7)}$	$\sim$
X_2	$X_2^{(1)}$	$X_2^{(2)}$	$X_2^{(3)}$	$X_2^{(4)}$	$X_2^{(5)}$	$X_2^{(6)}$	$X_2^{(7)}$	$\sim$
X_3	$X_3^{(1)}$	$X_3^{(2)}$	$X_3^{(3)}$	$X_4^{(4)}$	$X_5^{(5)}$	$X_6^{(6)}$	$X_7^{(7)}$	$\sim$
$\{$								

we run resampling algo and $\text{check}(T)$ using same samples

claim: if $\log$ induces T , then $\text{check}(T)$ succeed

Resampling algorithm (for general LL) (Moser-Tardos, 2010)

1. Sample all variables V (independently, from their respective distributions)
2. while there is an event A_i :
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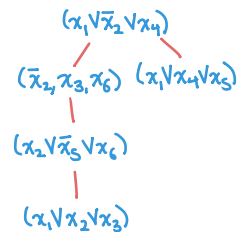
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Suppose first i events are $A_{i_1}, A_{i_2}, \sim, A_{i_i}$.

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check(T)

- ① For each event A_{i_j} in T in decr. depth: (leaf first, root last)
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we run resampling algo and check(T) using same samples

claim: if log induces T, then check(T) succeeds

V	1st sample	2nd sample	3rd sample	4th sample	5th	6th	7th	
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$\{$								

each A_{ij} maps to sample of S_{ij} inducing A_{ij} (before resampling A_{ij})

say log = $A_{i1}, A_{i2}, A_{i3}, \dots$

say T has labels $A_{j1}, A_{j2}, A_{j3}, \dots$ in decr. depth

we need each A_{jk} to use

sample of S_{jk} inducing A_{jk}

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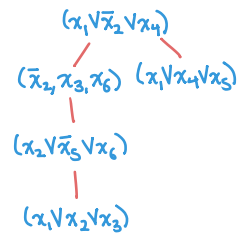
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check(T)

- ① For each event A_{ij} in T in decr. depth: (label) (leaf first, root last)
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we run resampling algo and check(T) using same samples

claim: if log induces T, then check(T) succeeds

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Fix node N in T w/ label A_i , corresponding to entry E

for $X_j \in S_i$, E uses sample $X_j^{(h)}$ where $h = \# \text{ entries before (and including) E w/ event using } X_j$

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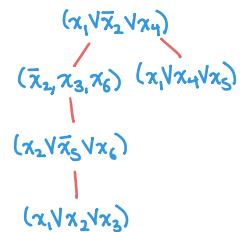
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Recall:

Suppose first i events are $A_{i_1}, A_{i_2}, \dots, A_{i_i}$.

① Make root w/ label A_{i_i} .

② For $j=i-1, i-2, \dots, 1$

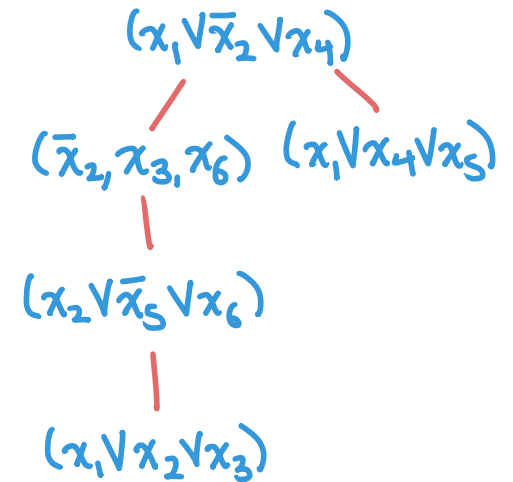
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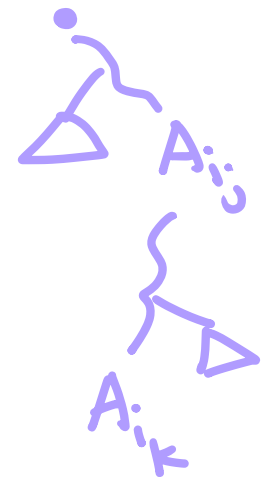


Observation (for later)

Let T_e = witness tree for $A_{i_1}, \dots, A_{i_e}$

Suppose $j < k$ and $A_{i_j} \sim A_{i_k}$

if A_{i_k} is a node in T_e , then so is A_{i_j} ,
 and A_{i_j} has greater depth than A_{i_k}



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claim: if log induces T, then check(T) succeeds

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Observation $\Rightarrow$

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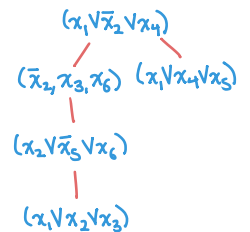
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claim: $\Pr[T_j = T \text{ for some } j] \leq \Pr[\text{check}(T)]$

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✓ claim: if log induces T,
(hence overall claim) then check(T) succeeds

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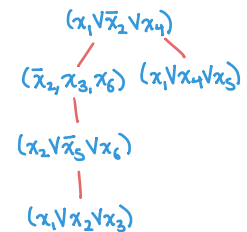
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Lemma $\Pr[T_j = T \text{ for some } j] \leq p(T) = \prod P[A_j]$
labels A_j in T
(w/ multiplicity)

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$$\Pr[\text{check}(T) \text{ succeeds}] = p(T) = \prod P[A_j]$$

labels A_j in T
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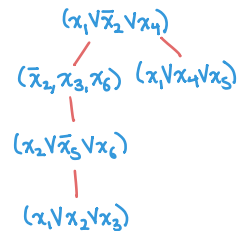
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let $W_i = \{\text{all witness trees w/ root } A_i\}$

for $j=1,2,\dots$, let T_j be witness tree for the first j entries of the log (or $\emptyset$ if log has $< j$ entries)

$$\checkmark E[\text{\# times } A_i \text{ resampled}] = \sum_{T \in W_i} P_r[T_j = T \text{ for some } j]$$

$$\checkmark P_r[T_j = T \text{ for some } j] \leq p(T) = \prod_{\text{labels } A_j \text{ in } T} P[A_j]$$

(w/ multiplicity)

claim: $\sum_{T \in W_i} p(T) \leq \frac{x(A)}{1-x(A)}$

(which completes the proof!)

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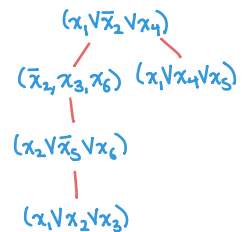
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Lemma: $\sum_{T \in \mathcal{W}_i} p(T) \leq \frac{x(A)}{1-x(A)}$ ($p(T) = \prod_{\substack{\text{labels } A_j \text{ in } T \\ (\text{w/ multiplicity})}} P[A_j]$)

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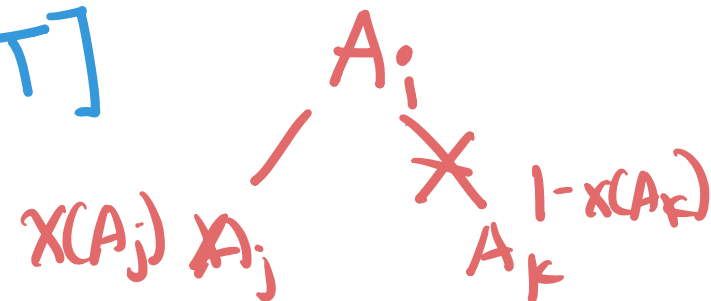
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Proof: Galton-Watson trees!

Round 1: Create root w/ label A_i .

Round $n+1$: $\forall$ nodes N created in prev. round, w/ label A_j , and each $A_k \sim A_j$, create child of N w/ label A_k w/ prob. $x(A_k)$

Fix $T \in \mathcal{W}_i$. let $q_T = P[\uparrow \text{ produces } T]$



Lemma: $\sum_{T \in \mathcal{W}_i} p(T) \leq \frac{x(A_i)}{1-x(A_i)}$

$(p(T) = \prod_{\substack{\text{labels } A_j \text{ in } T \\ (\text{w/ multiplicity})}} P[A_j])$

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Fix $T \in \mathcal{W}_i$. let $q_T = P[\text{produces } T]$

for node N in T , let
 $A_N = \text{event labeling } N$

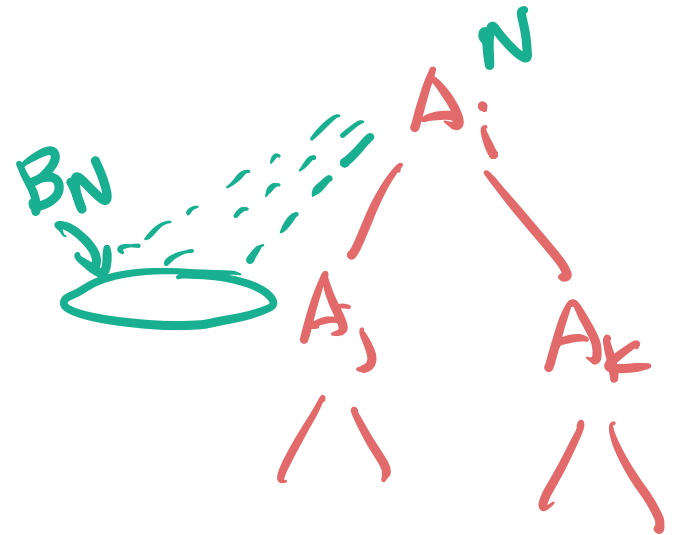
$B_N = \{A_k: A_k \sim A_N, \text{ but no child of } A_N \text{ labeled } A_k\}$

Round 1: Create root w/ label A_i .

Round $n+1$: $\forall$ nodes N created in prev. round, w/ label A_j , and each $A_k \sim A_j$, create child of N w/ label A_k w/ prob. $x(A_k)$

$$q_T = \frac{1}{x(A_i)} \prod_{N \in T} x(A_N) \prod_{A_k \in B_N} (1-x(A_k))$$

$$= \frac{1-x(A_i)}{x(A_i)} \underbrace{\prod_{N \in T} x(A_N)}_{P(T)} \prod_{\substack{A_k \sim A_N \\ A_k \neq A_N}} (1-x(A_k))$$



$$= p(T) \leq \prod_{N \in T} P[A_N] \leq \prod_{N \in T} x(A_N) \prod_{\substack{A_j \sim A_N \\ A_j \neq A_N}} (1-x(A_j))$$

$$= \frac{x(A_i)}{1-x(A_i)} q_T$$

$$\sum_T \mathbb{P}(T) \leq \frac{x(A_i)}{1-x(A_i)} \sum_{T \in \mathcal{T}_i} q_T$$

$$\leq \frac{x(A_i)}{1-x(A_i)}$$

Theorem: Suppose $\exists x(A_1), \dots, x(A_n) \in (0,1)$ s.t.
 $\mathbb{P}[A_i] \leq x(A_i) \prod_{\substack{A_j \sim A_i \\ A_j \neq A_i}} (1-x(A_j)) \quad \forall A_i.$

Then $\mathbb{P}[\bar{A}_1, \dots, \bar{A}_n] > 0$, and the resampling algo finds a configuration of $\mathcal{V}$ inducing $\bar{A}_1, \dots, \bar{A}_n$ while resampling each A_i at most $\frac{x(A_i)}{1-x(A_i)}$ times in expectation

Round 1: Create root w/ label A_i .

Round $n+1$: $\forall$ nodes N created in prev. round, w/ label A_j , and each $A_k \sim A_j$, create child of N w/ label A_k w/ prob. $x(A_k)$

✓
$$q_T = \frac{1 - \chi(A_i)}{\chi(A_i)} \prod_{N \in T} \chi(A_N) \prod_{\substack{A_j \sim A_N \\ A_j \neq A_N}} (1 - \chi(A_j))$$

$$P(T) = \prod_{\substack{\text{labels } A_j \text{ in } T \\ (\text{w/ multiplicity})}} P[A_j]$$

Theorem: Suppose $\exists \chi(A_i), \sim, \chi(A_n) \in (0,1)$ s.t.
 $P[A_i] \leq \chi(A_i) \prod_{\substack{A_j \sim A_i \\ A_j \neq A_i}} (1 - \chi(A_j)) \quad \forall A_i.$

Then $P[\bar{A}_i, \sim, \bar{A}_n] > 0$, and the resampling algo finds a configuration of V inducing $\bar{A}_i, \sim, \bar{A}_n$ while resampling each A_i at most $\frac{\chi(A_i)}{1 - \chi(A_i)}$ times in expectation

Round 1: Create root w/ label A_i .

Round $n+1$: $\forall$ nodes N created in prev. round, w/ label A_j , and each $A_k \sim A_j$, create child of N w/ label A_k w/ prob. $\chi(A_k)$

Theorem: Suppose $\exists \chi(A_1), \dots, \chi(A_n) \in (0,1)$ s.t.

$$P[A_i] \leq \chi(A_i) \prod_{\substack{A_j \sim A_i \\ A_j \neq A_i}} (1 - \chi(A_j)) \quad \forall A_i.$$

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Round 1: Create root w/ label A_i .

Round $n+1$: $\forall$ nodes N created in prev. round, w/ label A_j , and each $A_k \sim A_j$, create child of N w/ label A_k w/ prob. $\chi(A_k)$

let $W_i = \{\text{all witness trees w/ root } A_i\}$

for $j=1,2,\dots$, let T_j be witness tree for the first j entries of the log (or $\emptyset$ if log has $< j$ entries)

$$\checkmark E[\text{\# times } A_i \text{ resampled}] = \sum_{T \in W_i} P_r[T_j = T \text{ for some } j]$$

$$\checkmark P_r[T_j = T \text{ for some } j] \leq p(T) = \prod_{\text{labels } A_j \text{ in } T} P[A_j]$$

(w/ multiplicity)

$$\checkmark \text{claim: } \sum_{T \in W_i} p(T) \leq \frac{x(A)}{1-x(A)}$$

(which completes the proof!)

Resampling algorithm (for general LL) (Moser-Tardos, 2010)

1. Sample all variables V (independently, from their respective distributions)
2. while there is an event A_i :
 - a. resample all of S_i (independently, from their respective distributions)

Theorem: Suppose $\exists x(A_1), \dots, x(A_n) \in (0,1)$ s.t.
 $P[A_i] \leq x(A_i) \prod_{\substack{A_j \sim A_i \\ A_j \neq A_i}} (1-x(A_j)) \quad \forall A_i$

Then $P[\bar{A}_1, \dots, \bar{A}_n] > 0$, and the resampling algo finds a configuration of V inducing $\bar{A}_1, \dots, \bar{A}_n$ while resampling each A_i at most $\frac{x(A_i)}{1-x(A_i)}$ times in expectation

Mapping prefixes of the log $\rightarrow$ witness trees

Suppose first i events are $A_{i_1}, A_{i_2}, \dots, A_{i_i}$.

① Make root w/ label A_{i_1} .

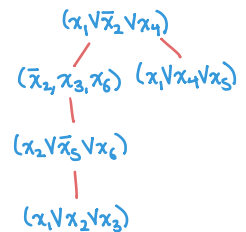
② For $j=i-1, i-2, \dots, 1$

ⓐ If $A_{i_j} \sim A_k$ for some label A_k in the tree

① pick such a node N of max depth

ⓐ make a new child of N w/ label A_{i_j}

Log
 $(x_1 \vee x_2 \vee x_3), (x_2 \vee \bar{x}_5 \vee x_6), (x_1, \bar{x}_8, x_9),$
 $(x_1 \vee x_4 \vee x_5), (\bar{x}_2, x_3, x_6), (x_1 \vee \bar{x}_2 \vee x_4)$



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