

Conductance + Mixing Time

Lecture 16

Probability (and techniques)

Algorithms (models and applications)

Random Walks

Stationary Distribution

Perron-Frobenius Theorem

Pagerank

Lecture 18

Probability (and techniques)

Algorithms (models and applications)

Hitting times

Cover times

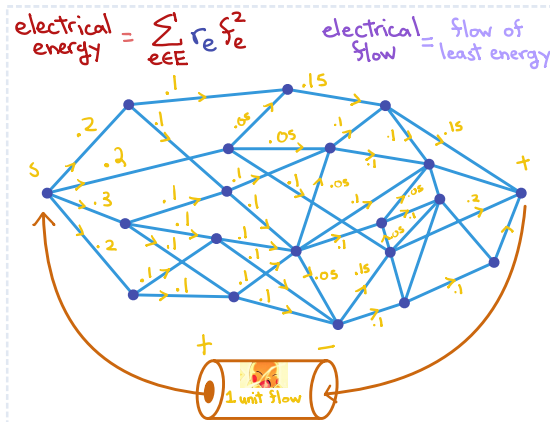
Electrical networks

Connectivity
in $O(\log n)$ space

random walk terminology:

Hitting time $H(s, t)$ = expected # steps until a random walk from s reaches t

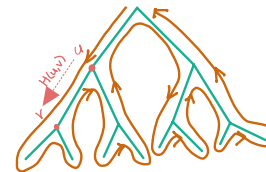
Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex



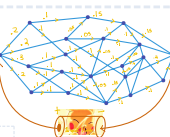
Lemma $H(u, v) + H(v, u) = 2m \cdot (\text{energy of unit flow } (u, v))$

Theorem $C(s) \leq 2mn$

proof Six traversal of spanning tree



$$\begin{aligned} \text{total time} &= \sum_{s, t \in V} H(u, v) + H(v, u) \\ &= 2m \sum_{s, t \in V} (\text{energy of unit flow } (u, v)) \leq 2m(n-1) \end{aligned}$$



electrical energy = $\sum_{e \in E} r_e f_e^2$
electrical flow = least energy

des $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
(BS) $_v$ = net flow at v
 $d_v = \begin{cases} -1 & \text{if } v = s \\ 1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}$

min $\frac{1}{2} \sum_{e \in E} r_e f_e^2$ s.t. $BS = d$
(electrical energy) (is a typical)

effective resistance of d = optimum value/energy
Ohm's law: $f = R^{-1} B^T p$
for electric flow by electric potentials p
 $L = BR^{-1} B^T \in \mathbb{R}^{V \times V}$
(effective resist. of d) $= \langle d, L^{-1} d \rangle$

Lecture 19

Probability (and techniques)

Algorithms (models and applications)

Spectral theorem
for symmetric maps

Spectral analysis of
random walks

Let R have eigenvalues

$$\mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n \geq -1$$

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

Lecture 19

Probability (and techniques)

Algorithms (models and applications)

Spectral theorem
for symmetric maps

Spectral analysis of
random walks

Lecture 20

Probability
(and techniques)

Algorithms
(models and applications)

Conductance

Cheeger's inequality

Conductance-based
mixing time

Fiedler's algorithm
for low-conductance
cuts

Sparsest cut

Input: undirected graph $G=(V,E)$
edge capacities $c(e) > 0$ for $e \in E$
demands $b(u,v) \geq 0$ for $u,v \in V$ (symmetric)

Goal: compute SCV minimizing

"sparsity" $\frac{c(\delta(S))}{\sum_{u \in S, v \in \bar{S}} b(u,v)}$

← cut few edges

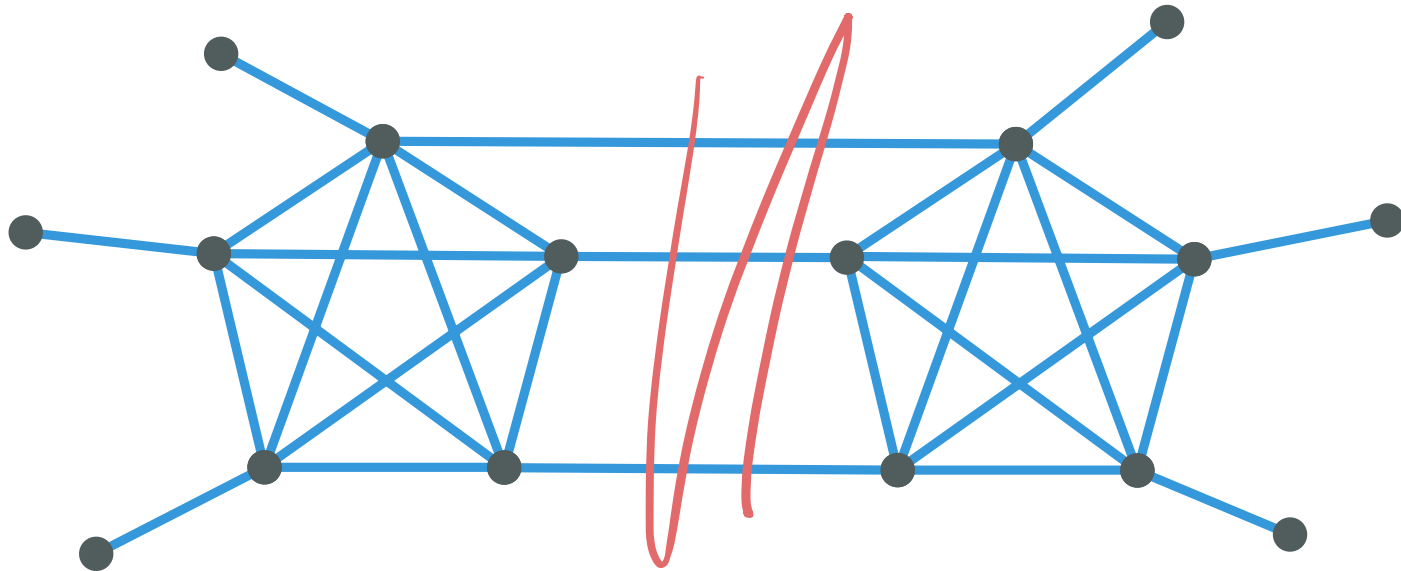
← split up many pairs

" $\delta(S)$ " = edges cut by S

Special case: uniform sparsest cut

w/ uniform demands $b(u,v)=1 \quad \forall u,v$

$$\text{"sparsity"} = \frac{c(\delta(S))}{|S| |\bar{S}|}$$



useful primitive for graph partitioning, divide + conquer

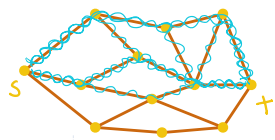
Lecture 10

Probability (and techniques)

LP duality

L_1 -metric embeddings

Max Flow Min Cut



Path Packing LP

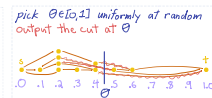
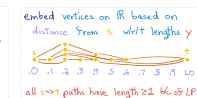
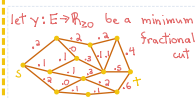
maximize $\sum_{p \in P} x_p$
over $x: P_{st} \rightarrow \mathbb{R}_{\geq 0}$
(all s-t paths)

s.t. $\sum_{p \in P} x_p \leq c_e$ for all edges e
"Ax ≤ c" (capacity of e)

Path Covering LP (i.e. cut)

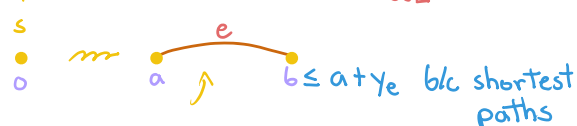
minimize $\sum_{e \in E} c_e y_e$
over $y: E \rightarrow \mathbb{R}_{\geq 0}$
(total capacity of "fractional cut")

s.t. $\sum_{e \in P} y_e \geq 1$ for all $p \in P_{st}$
"A'y ≥ b" (continuous analog of s-t cut)



$E[\text{capacity of cut}]$

$$\stackrel{\text{(linearity of expectation)}}{=} \sum_{e \in E} c_e \Pr[e \text{ cut}] \leq \sum_{e \in E} c_e y_e$$

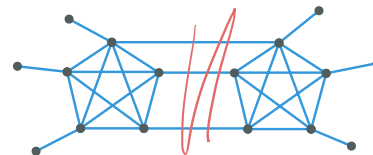


Algorithms

Sparsest Cut

Special case: uniform sparsest cut
w/ uniform demands $b(u,v) = 1 \forall u,v$

$$\text{"sparsity"} = \frac{c(\delta(s))}{|S| |T|}$$



useful primitive for graph partitioning, divide & conquer

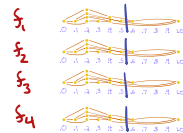
Q: how to round LP to sparse cut?

min $\sum_{e \in E} c_e d(u,v)$ over metrics $d: V \times V \rightarrow \mathbb{R}_{\geq 0}$
s.t. $\sum_{i=1}^k b_i d(s_i, t_i) \geq 1$

Suppose $d(u,v)$ was of the form

$$d(u,v) = \sum_{i=1}^k |f_i(u) - f_i(v)| \text{ for } f_i: V \rightarrow [0,1]^k$$

(assume $\min f_i(a) = 0, \max f_i(a) = 1$)



" L_1 -metric"

Setup / notation

undirected graph $G=(V,E)$

edge weights $w: E \rightarrow \mathbb{R}_{>0}$

for $S \subset V$, " $\delta(S)$ " = set of edges cut by S

" $w(\delta(S))$ " = weight of edges cut by S

for $S \subset V$, " $\text{vol}(S)$ " = $\sum_{v \in S} \deg(v)$

$\bar{S} = V \setminus S$

Conductance

$G=(V,E)$ undirected
edge weights $w: E \rightarrow \mathbb{R}_{\geq 0}$

" $\delta(S)$ " = edges cut by S

" $\text{vol}(S)$ " = $\sum_{v \in S} \deg(v)$

"conductance of S "

$$\text{for } S \subset V, \quad \psi(S) = \frac{w(\delta(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

"conductance of G "

$$\psi(G) = \min_{\emptyset \subsetneq S \subsetneq V} \psi(S)$$

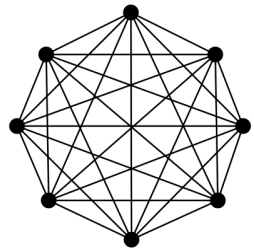
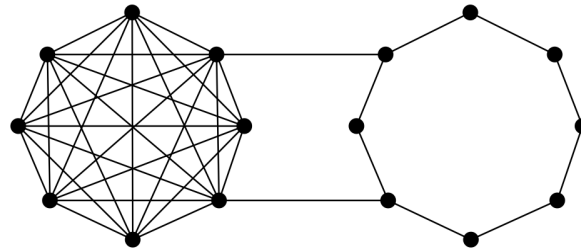
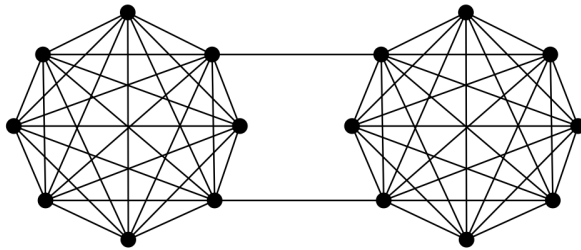
Sparsity

$$\frac{w(\delta(S))}{\min\{|S|, |\bar{S}|\}}$$

Conductance

$$\frac{w(\delta(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

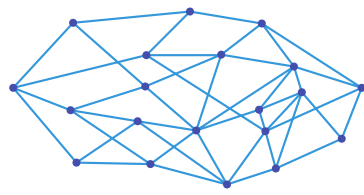
(high/medium/low)



Sparsity

conductance

Laplacian of



$G=(V,E)$ w/ edge weights $w \in \mathbb{R}_{\geq 0}^E$

$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{\substack{e \in E \\ e=\{u,v\}}} w(e) (x_u - x_v)^2$$

let $D = \text{diag}(d)$ where $d \in \mathbb{R}^V$ are degrees

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

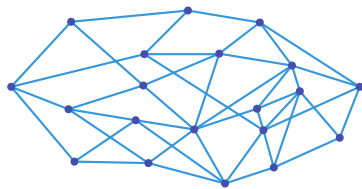
Laplacian of $e = \{u, v\}$

$$L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{V \times V}$$

($1_u = \{0,1\}$ -indicator vector for u)

$$\langle x, L_e x \rangle = (x_u - x_v)^2$$

Laplacian of



$G=(V,E)$ w/ edge weights $w \in \mathbb{R}_{\geq 0}^E$

$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{\substack{e \in E \\ e=\{u,v\}}} w(e) (x_u - x_v)^2$$

let $D = \text{diag}(d)$ where $d \in \mathbb{R}^V$ are degrees

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$
Laplacian of $e = \{u, v\}$
 $L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{V \times V}$
($1_u = \{0,1\}$ -indicator vector for u)
 $\langle x, L_e x \rangle = (x_u - x_v)^2$

also, $L = D - A$

$G = (V, E)$ undirected
edge weights $w: E \rightarrow \mathbb{R}_{\geq 0}$

" $\delta(S)$ " = edges cut by S

" $\text{vol}(S)$ " = $\sum_{v \in S} \deg(v)$

$$\psi(S) = \frac{w(\delta(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\psi(G) = \min_{\emptyset \subsetneq S \subsetneq V} \psi(S)$$

Laplacian of 

$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

$$D = \text{diag}(\text{degrees})$$

if $\text{vol}(S) \leq \text{vol}(\bar{S})$,

$$\psi(S) = \frac{w(\delta(S))}{\text{vol}(S)} = \frac{\langle \mathbf{1}_S, L \mathbf{1}_S \rangle}{\langle \mathbf{1}_S, D \mathbf{1}_S \rangle}$$

define normalized Laplacian

$$M = D^{-1/2} L D^{-1/2}$$

$$\frac{\langle x, Lx \rangle}{\langle x, Dx \rangle} = \frac{\langle y, My \rangle}{\langle y, y \rangle} \quad \text{for } y = D^{1/2} x$$

$G=(V,E)$ undirected
edge weights $w: E \rightarrow \mathbb{R}_{\geq 0}$

" $\delta(S)$ " = edges cut by S

" $\text{vol}(S)$ " = $\sum_{v \in S} \deg(v)$

$$\psi(S) = \frac{w(\delta(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\psi(G) = \min_{\emptyset \subsetneq S \subsetneq V} \psi(S)$$

Laplacian of 
 $L = \sum_{e \in E} w(e) L_e$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

$D = \text{diag}(\text{degrees})$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ max/min $\langle x, Lx \rangle$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$
where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

Normalized random walk matrix

$$Q = D^{-1/2} R D^{1/2} = D^{-1/2} A D^{-1/2}$$

- $D = \text{diag}(d)$ w/ $d = \text{degrees}$
- $A = \text{adjacency matrix}$
- $R = A D^{-1} = \text{random walk matrix}$

Q is symmetric!

Q has same eigenvalues as R !

$$Q = v_1 \otimes v_1 + \mu_2 (v_2 \otimes v_2) + \dots + \mu_n (v_n \otimes v_n)$$

$$\text{w/ } \mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n$$

(also eigenvalues of R)

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ maximizes $\langle x, Lx \rangle$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$
Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$
where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

Theorem: $G = (V, E)$ undirected, $w: E \rightarrow \mathbb{R}_{>0}$
then R has stationary distr. $s \in \Delta_V$ w/

$$s_v = \frac{\deg(v)}{2W} \quad \deg(v) = \sum_{e \in \partial(v)} w(e), \quad W = \sum_{e \in E} w(e)$$

Lemma let $A = C^{-1} B C$.

- A and B have same eigenvalues
- C maps eigenvectors of A to eigenvectors of B

$$Q = \frac{\sqrt{d} \otimes \sqrt{d}}{\langle \mathbf{1}, d \rangle} + \mu_2 (v_2 \otimes v_2) + \dots + \mu_n (v_n \otimes v_n)$$

w/ $\mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n$
(also eigenvalues of R)

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary distribution

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ maximizes $\langle x, Lx \rangle$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$
where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

Theorem: $G=(V, E)$ undirected, $w: E \rightarrow \mathbb{R}_{>0}$
then R has stationary distr. $s \in \Delta_V$ w/

$$s_v = \frac{\deg(v)}{2W} \quad \deg(v) = \sum_{e \in \mathcal{E}(v)} w(e), \quad W = \sum_{e \in E} w(e)$$

Lemma let $A = C^{-1}BC$.

- A and B have same eigenvalues
- C maps eigenvectors of A to eigenvectors of B

Normalized random walk matrix

$$Q = D^{-1/2} R D^{1/2} = D^{-1/2} A D^{1/2}$$

- $D = \text{diag}(d)$ w/ $d = \text{degrees}$
- $A = \text{adjacency matrix}$
- $R = A D^{-1} = \text{random walk matrix}$
- Q is symmetric!
- Q has same eigenvalues as R !

normalized Laplacian $M = D^{-1/2} L D^{-1/2}$

Normalized random walk matrix

$$Q = D^{-1/2} R D^{1/2} = D^{-1/2} A D^{-1/2}$$

- $D = \text{diag}(d)$ w/ $d = \text{degrees}$
- $A = \text{adjacency matrix}$
- $R = A D^{-1} = \text{random walk matrix}$

$$M = D^{-1/2} L D^{-1/2}$$

$$= D^{-1/2} (D - A) D^{-1/2}$$

$$= I - Q = D^{-1/2} (I - R) D^{1/2}$$

$G = (V, E)$ undirected
edge weights $w: E \rightarrow \mathbb{R}_{\geq 0}$

" $\delta(S)$ " = edges cut by S

$$\text{"vol}(S) = \sum_{v \in S} \deg(v)$$

$$\psi(S) = \frac{w(\delta(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\psi(G) = \min_{\emptyset \subset S \subset V} \psi(S)$$

Laplacian of 
 $L = \sum_{e \in E} w(e) L_e$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

$$D = \text{diag}(\text{degrees})$$

$$L = D - A$$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ max/min $\langle x, Lx \rangle$ over $\|x\|=1$
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Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$
where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

if $\text{vol}(S) \leq \text{vol}(\bar{S})$,

$$\psi(S) = \frac{w(\delta(S))}{\text{vol}(S)} = \frac{\langle \mathbf{1}_S, L \mathbf{1}_S \rangle}{\langle \mathbf{1}_S, D \mathbf{1}_S \rangle}$$

normalized Laplacian $M = D^{-1/2} L D^{-1/2}$

$$\checkmark M = I - Q = D^{-1/2} (I - R) D^{1/2}$$

Theorem: M has same eigenvalues as $I - R$.

$$\lambda \text{ for } M \rightarrow \lambda - 1 \text{ for } R$$

eigenvalue eigenvalue

$$\lambda \text{ for } R \rightarrow 1 - \lambda \text{ for } M$$

eigenvalue eigenvalue

e.g. smallest eigenvalue of M is

$G = (V, E)$ undirected
edge weights $w: E \rightarrow \mathbb{R}_{\geq 0}$

" $\delta(S)$ " = edges cut by S

$$\text{"vol}(S) = \sum_{v \in S} \deg(v)$$

$$\Psi(S) = \frac{w(\delta(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\Psi(G) = \min_{\emptyset \subsetneq S \subsetneq V} \Psi(S)$$

Laplacian of 
 $L = \sum_{e \in E} w(e) L_e$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

$$D = \text{diag}(\text{degrees})$$

$$L = D - A$$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ max/min $\langle x, Lx \rangle$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$
where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

if $\text{vol}(S) \leq \text{vol}(\bar{S})$,

$$\Psi(S) = \frac{w(\delta(S))}{\text{vol}(S)} = \frac{\langle \mathbf{1}_S, L \mathbf{1}_S \rangle}{\langle \mathbf{1}_S, D \mathbf{1}_S \rangle}$$

Cheeger's inequality:

let λ_2 = 2nd smallest eigenvalue of M

then
$$\frac{\lambda_2}{2} \leq \Psi(G) \leq \sqrt{2\lambda_2}$$

- connects combinatorial properties of G w/ algebraic properties

$$\Psi(S) = \frac{w(S(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\Psi(G) = \min_{\emptyset \subset S \subset V} \Psi(S)$$

Laplacian of 

$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

$$D = \text{diag}(\text{degrees})$$

$$L = D - A$$

normalized Laplacian

$$M = D^{-1/2} L D^{-1/2}$$

Normalized random walk matrix

$$Q = D^{-1/2} R D^{1/2} = D^{-1/2} A D^{-1/2}$$

- $D = \text{diag}(d)$ w/ $d = \text{degrees}$
- $A = \text{adjacency matrix}$
- $R = A D^{-1} = \text{random walk matrix}$

Theorem: M has same eigenvalues as $I - R$.

$$\lambda \text{ for } M \rightarrow \lambda - 1 \text{ for } R$$

eigenvalue eigenvalue

$$\lambda \text{ for } R \rightarrow 1 - \lambda \text{ for } M$$

eigenvalue eigenvalue

1

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

$1 \geq \mu_2 \geq \dots \geq \mu_n = \text{eigenvalues of } R$

2

Theorem: M has same eigenvalues as $I - R$.

λ for $M \rightarrow \lambda - 1$ for R
eigenvalue eigenvalue

λ for $R \rightarrow 1 - \lambda$ for M
eigenvalue eigenvalue

3

Cheeger's inequality:

$$\frac{\lambda_2}{2} \leq \psi(G) \leq \sqrt{2\lambda_2}$$

$\lambda_2 = 2\text{nd smallest eigenvalue of } M$

let $\Psi = \Psi(G)$

Lemma: if $\mu_2 \geq |\mu_n|$, then

$$\|R^t x - s\| \leq e^{-t\Psi/4} \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

proof:

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

$1 \geq \mu_2 \geq \dots \geq \mu_n = \text{eigenvalues of } R$

Theorem: M has same eigenvalues as $I - R$.

λ for $M \rightarrow \lambda - 1$ for R
eigenvalue eigenvalue

λ for $R \rightarrow 1 - \lambda$ for M
eigenvalue eigenvalue

Cheeger's inequality:

$$\frac{\lambda_2}{2} \leq \Psi(G) \leq \sqrt{2\lambda_2}$$

$\lambda_2 = 2\text{nd smallest eigenvalue of } M$

let $\Psi = \Psi(G)$

✓ Lemma: if $\mu_2 \geq |\mu_n|$, then

$$\|R^t x - s\| \leq e^{-t\Psi^2/4} \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

Q: what if $\mu_2 < |\mu_n|$?

Or: how to force μ_n away from -1 ?

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

$1 \geq \mu_2 \geq \dots \geq \mu_n = \text{eigenvalues of } R$

Theorem: M has same eigenvalues as $I - R$.

λ for $M \rightarrow \lambda - 1$ for R
eigenvalue eigenvalue

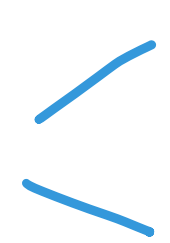
λ for $R \rightarrow 1 - \lambda$ for M
eigenvalue eigenvalue

Cheeger's inequality:

$$\frac{\lambda_2}{2} \leq \Psi(G) \leq \sqrt{2\lambda_2}$$

$\lambda_2 = 2\text{nd smallest eigenvalue of } M$

Lazy random walk

w/ prob. $\frac{1}{2}$:  stay at same vertex
take random step

let $S \in [0,1]^{n \times n}$ be lazy random walk matrix

$$S = \frac{1}{2}I + \frac{1}{2}R$$

note: S has same stationary distribution as R

let $\Psi = \Psi(G)$

Theorem:

$$\|S^\dagger x - s\| \leq e^{-\Psi^2/4} \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

proof:

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

$1 \geq \mu_2 \geq \dots \geq \mu_n = \text{eigenvalues of } R$

Theorem: M has same eigenvalues as $I - R$.

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Cheeger's inequality:

let λ_2 = 2nd smallest eigenvalue of M
then

exercise

$$\frac{\lambda_2}{2} \leq \underbrace{\Psi(G)}_{\text{algorithmic, rest of class}} \leq \sqrt{2\lambda_2}$$

algorithmic, rest of class

- connects combinatorial properties of G w/ algebraic properties

$$\Psi(S) = \frac{w(S(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\Psi(G) = \min_{\emptyset \subsetneq S \subsetneq V} \Psi(S)$$

Laplacian of 

$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

$$D = \text{diag}(\text{degrees})$$

$$L = D - A$$

normalized Laplacian

$$M = D^{-1/2} L D^{-1/2}$$

$$\Psi(G) \leq \sqrt{2\lambda_2}$$

high level ideas:

$$M = (\lambda_1=0)(v_1 \otimes v_1) + \lambda_2(v_2 \otimes v_2) + \dots + \lambda_n(v_n \otimes v_n)$$

$v_1, \dots, v_n$ orthonormal, $\lambda_1=0 \leq \lambda_2 \leq \dots \leq \lambda_n$

$v_1 =$ eigenvector for eigenvalue 1 of R
 $= \sqrt{d}/\langle \mathbf{1}, d \rangle$

Theorem: M has same eigenvalues as $I-R$.
 λ for $M \rightarrow \lambda-1$ for R
eigenvalue eigenvalue
 λ for $R \rightarrow 1-\lambda$ for M
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$$\lambda_2 = \langle v_2, M v_2 \rangle = \min \frac{\langle y, M y \rangle}{\langle y, y \rangle} \text{ s.t. } \langle y, \sqrt{d} \rangle = 0$$

$$= \min \frac{\langle x, L x \rangle}{\langle x, D x \rangle} \text{ s.t. } \langle x, d \rangle = 0$$

$$\Psi(G) \leq \sqrt{2\lambda_2}$$

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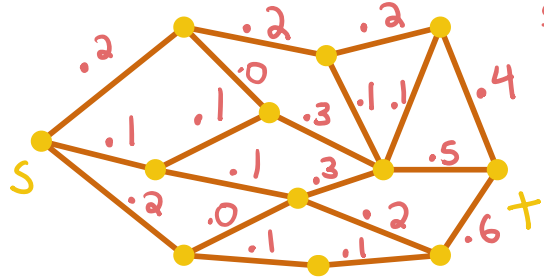
goal: convert x to $S \subseteq V$ s.t.

$$\Psi(S) \leq \sqrt{\frac{\langle x, Lx \rangle}{\langle x, Dx \rangle}} = \sqrt{\lambda_2}$$

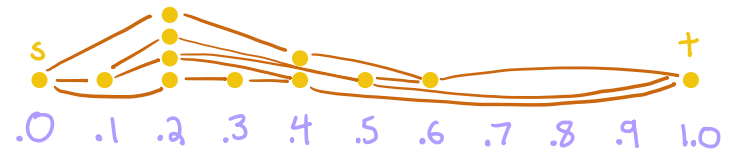
how?

Recall:

let $\gamma: E \rightarrow \mathbb{R}_{\geq 0}$ be a minimum fractional cut



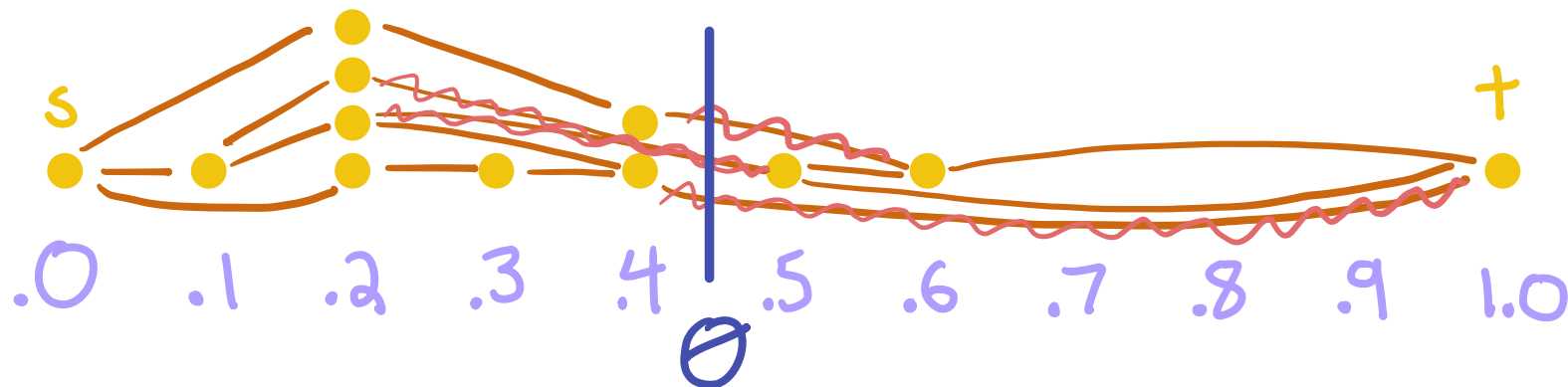
embed vertices on $\mathbb{R}$ based on distance from s w/r/t lengths γ



all $s \rightsquigarrow t$ paths have length ≥ 1 b/c of LP

pick $\theta \in [0, 1]$ uniformly at random

output the cut at θ



$$\Psi(G) \leq \sqrt{2\lambda_2}$$

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high level ideas:

$$\lambda_2 = \langle v_2, M v_2 \rangle = \min \frac{\langle x, Lx \rangle}{\langle x, Dx \rangle} \text{ s.t. } \langle x, d \rangle = 0$$

claim: if $\langle x, d \rangle = 0$, line embedding of x induces

$$\text{cut } S \text{ s.t. } \Psi(S) = \sqrt{2 \frac{\langle x, Lx \rangle}{\langle x, Dx \rangle}}$$

let $V = \sum_v \deg(v) = \text{total volume}$

Lemma: let $x \in \mathbb{R}_{\geq 0}^V$ s.t. $\text{vol}(\text{support}(x)) \leq \frac{V}{2}$

then $\langle x, Lx \rangle \geq \frac{\psi^2}{2} \langle x, Dx \rangle$

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Proof. let $S = \text{support}(x) = \{v_1, \dots, v_k\}$, $x(v_1) \geq x(v_2) \geq \dots \geq x(v_k)$
 $S_i = \{v_1, \dots, v_i\}$ for each i

$$\begin{aligned} \frac{\Psi}{2} \langle x, Dx \rangle &= \Psi \sum_{i=1}^k x^2(v_i) \deg(v_i) \\ &= \Psi \sum_{i=1}^k x^2(v_i) (\text{vol}(S_i) - \text{vol}(S_{i-1})) \\ &= \Psi \sum_{i=1}^k \text{vol}(S_i) (x^2(v_i) - x^2(v_{i+1})) \end{aligned}$$

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Lemma: let $\langle x, d \rangle = 0$. then $\forall y = x + d \mathbb{1}$,

$$\frac{\langle y, Ly \rangle}{\langle y, Dy \rangle} \leq \frac{\langle x, Lx \rangle}{\langle x, Dx \rangle}$$

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Proof.

$$\langle y, Ly \rangle = \langle x, Lx \rangle \quad \leftarrow \text{why?}$$

$$\langle y, Dy \rangle = \langle x + d\mathbb{1}, D(x + d\mathbb{1}) \rangle$$

$$= \langle x, x \rangle + d^2 \langle d, \mathbb{1} \rangle \quad \leftarrow \text{why?}$$

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Lemma: let $x = x_+ - x_-$ w/ $x_+, x_- \in \mathbb{R}_{\geq 0}^V$
 then

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Putting it all together:

$$\text{let } \lambda_2 = \frac{\langle x, Lx \rangle}{\langle x, Dx \rangle}, \quad \langle d, x \rangle = 0.$$

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