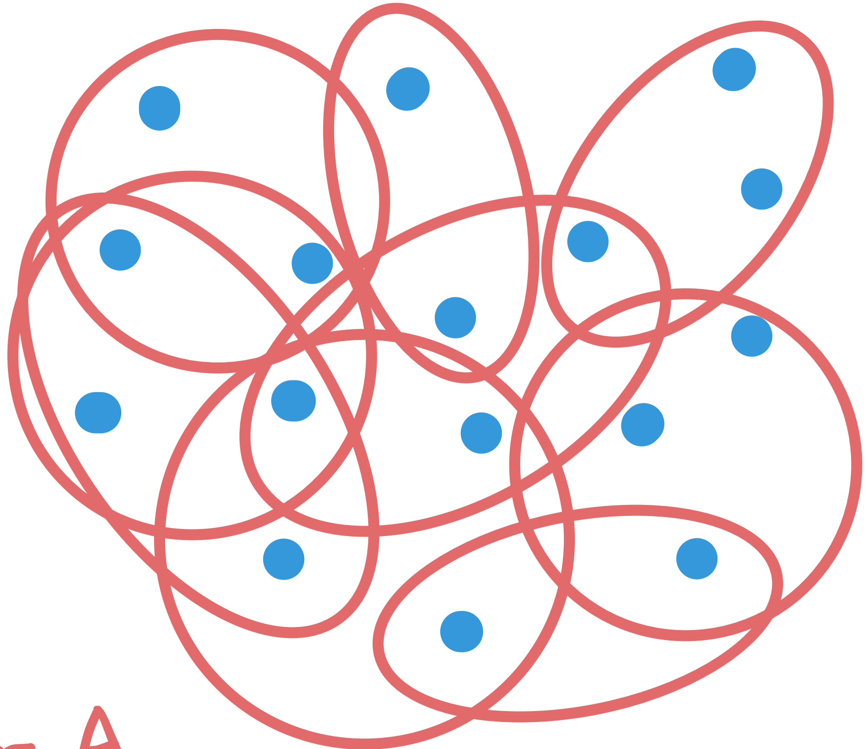
Randomized Greedy

Maximum Coverage



m points $[m] \equiv \{1, \dots, m\}$

n sets $A_1, \dots, A_n \subseteq [m]$

cardinality constraint $k \in \mathbb{N}$

Goal: select k sets $A_{e_1}, A_{e_2}, \dots, A_{e_k}$

maximizing the union $|A_{e_1} \cup A_{e_2} \cup \dots \cup A_{e_k}|$

NP-Hard via set cover

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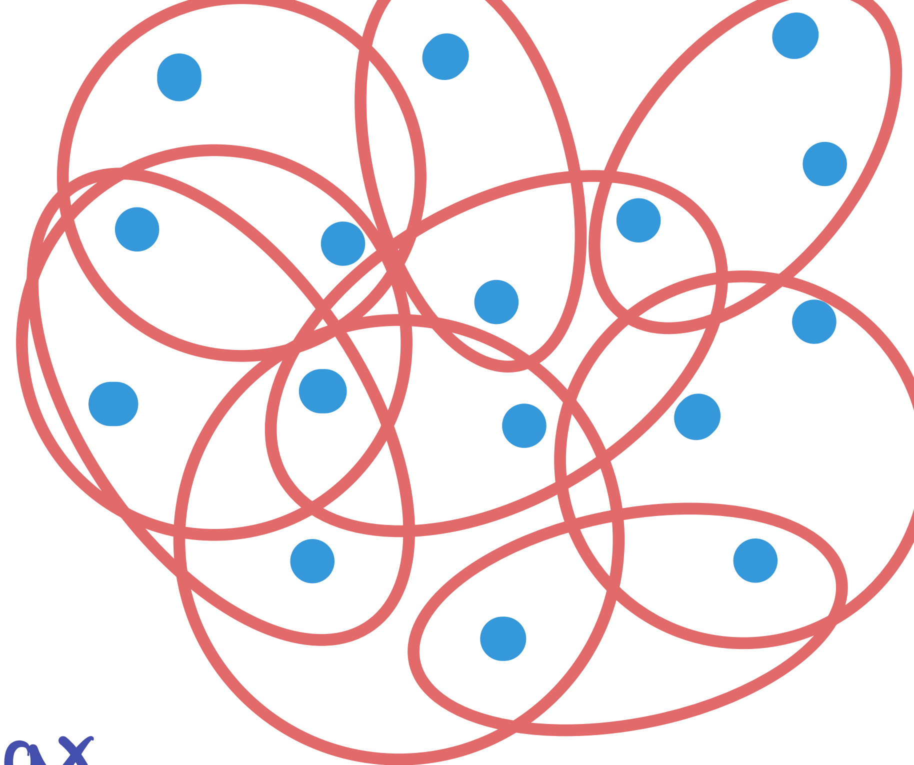
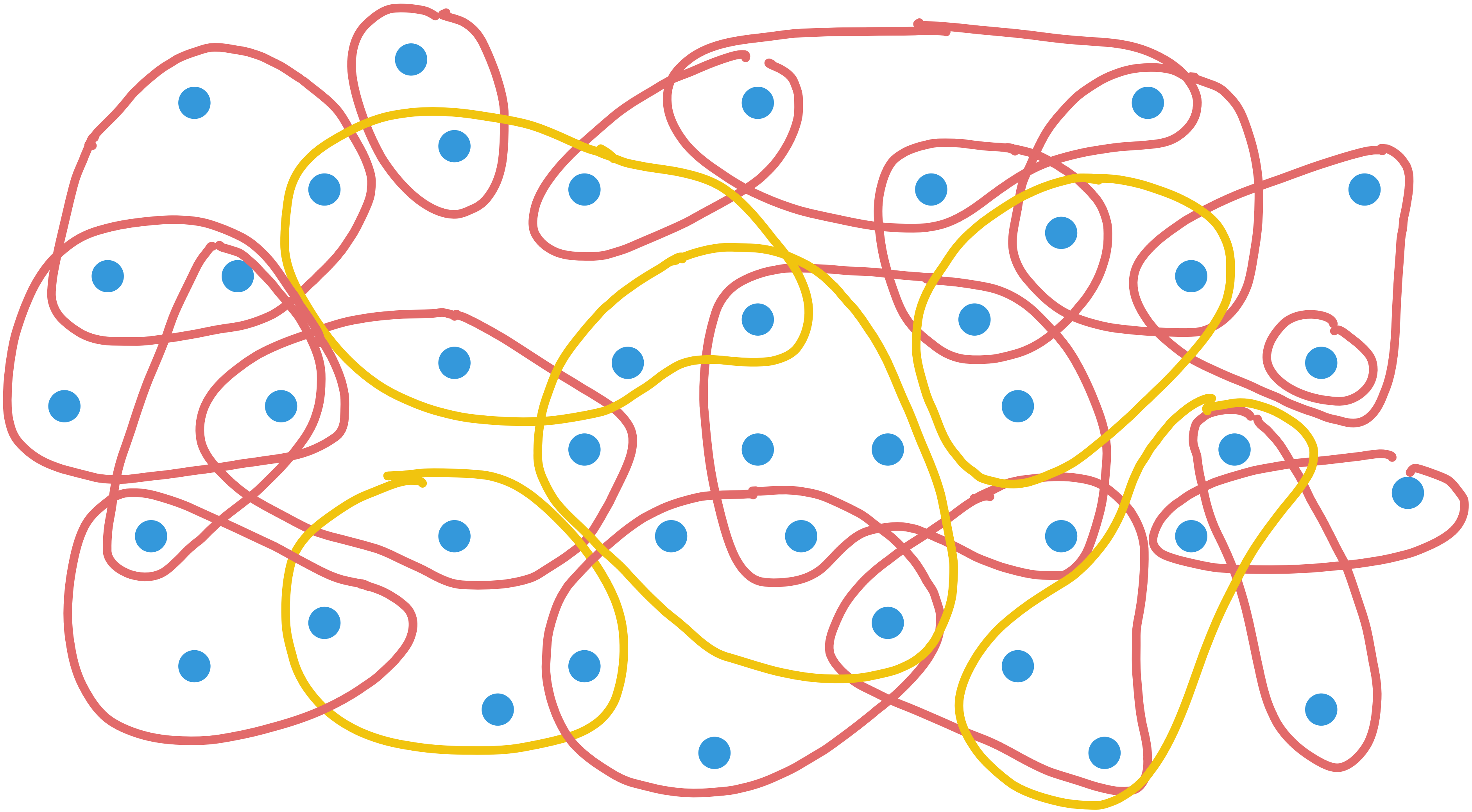


Set Cover (next time)	Max Coverage
minimize # sets to cover all points	maximize # points covered w/ $\leq k$ sets (packing constraint)
NP-Hard (via SAT) $\overset{\text{(why?)}}{\Longrightarrow}$	NP-Hard
LP + randomized rounding	greedy algorithm



greedy algorithm repeatedly selects the set covering the most new points

how good (approximation ratio) is the greedy algorithm?

Set function $f: 2^N \to \mathbb{R}_{\geq 0}$ s.t.

- **Normalized:** $f(\emptyset) = 0$
- **Nonnegative:** $f(S) \geq 0 \quad \forall S$
- **Monotone:** $f(S) \leq f(T)$ for $S \subseteq T$
- **Submodular:** for $A, B \subseteq N$,

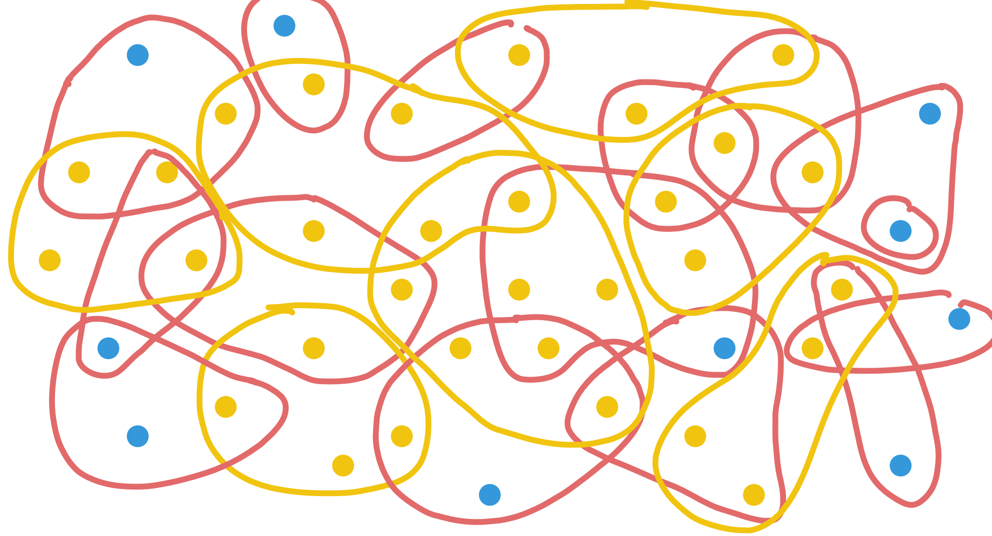
$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

equivalently:

"decreasing marginal values"

for $S \subseteq T \subseteq N$, $e \in N$,

$$\underbrace{f(S+e) - f(S)}_{\stackrel{\text{def}}{=} f(e|S)} \geq \underbrace{f(T+e) - f(T)}_{\stackrel{\text{def}}{=} f(e|T)}$$

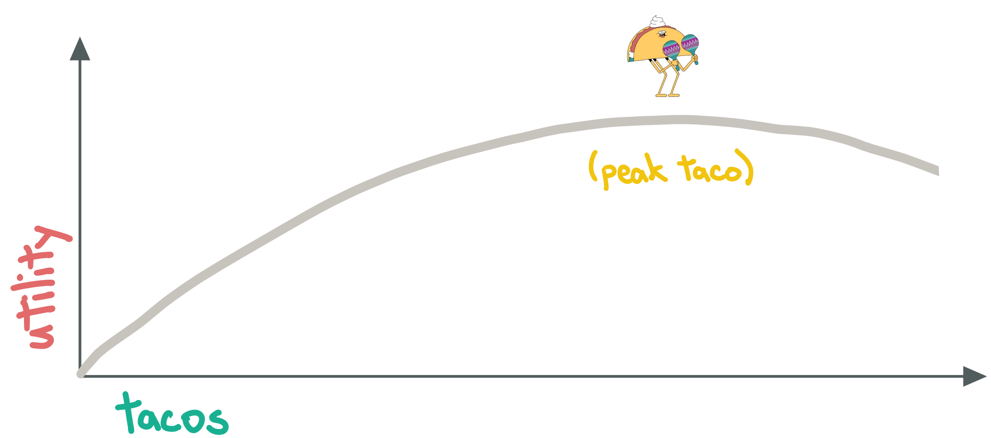


m points $[m] \equiv \{1, \ldots, m\}$

n sets $A_1, \ldots, A_n \subseteq [m]$

$S \subseteq [n]$ interpreted as collection of sets (via indices)

$$f(S) \stackrel{\text{def}}{=} \left(\begin{array}{c} \text{\# points covered by} \\ \text{sets (w/ index in) } S \end{array} \right)$$



greedy$(f, \mathcal{N}, k)$

1. $S_0 \leftarrow \emptyset$.
2. For $i = 1, \ldots, k$:
   A. Let $e_i \in \mathcal{N}$ maximize $f(e_i \mid S_{i-1})$
   B. $S_i \leftarrow S_i + e$.
3. Return S_k.

$|S^*| = k$

$f(S^*) = OPT$

Lemma $f(e_i \mid S_{i-1}) \geq \frac{1}{k}(OPT - f(S_{i-1}))$

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"decreasing marginal values"

for $S \subseteq T \subseteq \mathcal{N}$, $e \in \mathcal{N}$,

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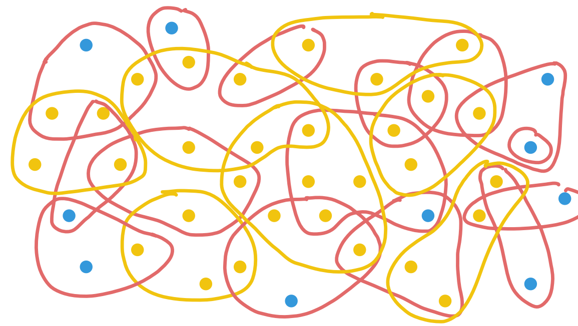

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Lemma $f(e_i \mid S_{i-1}) \geq \frac{1}{k}(OPT - f(S_{i-1}))$

$f(S^*) = OPT$

$$f(e_i \mid S_{i-1}) \geq \frac{1}{k}\sum_{j=1}^{k} f(d_j \mid \boxed{S_{i-1}})$$

$$\geq \frac{1}{k}\sum_{j=1}^{k} f(d_j \mid \boxed{S_{i-1} \cup S^*_{j-1}})$$

$S^* = \{d_1, \ldots, d_k\}$

$S^*_j = \{d_1, \ldots, d_j\}$

$$= \frac{1}{k}\sum_{j=1}^{k} f(S^*_j \cup S_{i-1}) - f(S^*_{j-1} \cup S_{i-1})$$

$$= \frac{1}{k}(f(S^* \cup S_{i-1}) - f(S_{i-1}))$$

$$\geq \frac{1}{k}(f(S^*) - f(S_{i-1}))$$

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Lemma $f(e_i \mid S_{i-1}) \geq \frac{1}{k}(OPT - f(S_{i-1}))$

Theorem $f(S_k) \geq (1-\frac{1}{e})OPT$

$$f(S_i) - f(S_{i-1}) \geq \frac{1}{k}(OPT - f(S_{i-1}))$$

$$\underbrace{OPT - f(S_i)}_{gap_i} \leq (1-\tfrac{1}{k})\underbrace{(OPT - f(S_{i-1}))}_{gap_{i-1}}$$

$$OPT - f(S_0) = OPT$$

$$OPT - f(S_1) \leq (1-\tfrac{1}{k})OPT$$

$$OPT - f(S_2) \leq (1-\tfrac{1}{k})^2 OPT \quad \Rightarrow \quad f(S_k) \geq (1-e^{-1})OPT$$

$$\vdots$$

$$OPT - f(S_k) \leq (1-\tfrac{1}{k})^k OPT \leq e^{-1} OPT$$

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Monotone Submodular f

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Suppose f was not monotone

does greedy work?

$\rightsquigarrow f(S^* | S_i) = f(S^* \cup S_i) - f(S_i)$

$\overset{\text{monotonicity}}{\geq} f(S^*) - f(S_i)$

f not monotone: $f(S^* \cup S_i) \not\geq f(S^*)$

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then

$$E[f(R)] \geq (1-p) f(\emptyset)$$

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e.g. if S_i random, $P[e \in S_i] \leq 1/2 \ \forall e$

$$E[f(S^* \cup S_i)] \geq \frac{1}{2} f(S^*)$$

$$g(S) = f(S \cup S^*)$$

Suppose f was not monotone

Lemma $R \subseteq \mathcal{N}$ random, $P[e \in R] \leq p$ $\forall e$
then $E[f(R)] \geq (1-p) f(\emptyset)$

Continuous extensions of f

$F : [0,1]^{\mathcal{N}} \to \mathbb{R}$ extends $f : 2^{\mathcal{N}} \to \mathbb{R}_{\geq 0}$

if $F(\mathbb{1}_S) = f(S)$ $\forall S \subseteq \mathcal{N}$

$F(x) = E[f(R)]$ R indep. sample
$\Pr[e \in R] = x_e$

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Convex closure F^-

$F^-(p) \overset{\text{def}}{=} \inf \{ E[f(R)] \text{ s.t. } P[e \in R] = p_e \, \forall e \}$

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Lovász:

$F^-(p) = E[f(S_t)]$ w/ $S_t = \{e : p_e \ge t\}$,
$t \in [0,1]$ unif. random

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$$F^-(p) \overset{\text{def}}{=} \inf_R \{E[f(R)] \text{ s.t. } P[e \in R] = p_e \ \forall e\}$$

Lovász: $F^-(p) = E[f(S_t)]$
w/ $S_t = \{e : p_e \geq t\}$, $t \in [0,1]$ unif. random

proof:

Suppose R has $P[e \in R] = p_e \ \forall e$

but $\exists\ A, B \subseteq \mathcal{N}$, s.t. $A \not\subseteq B$ $B \not\subseteq A$

$P[R=A] \geq P[R=B] = \delta > 0$

"T": increase $P[T = A\cup B]$ $P[T = A \cap B] \uparrow \delta$
decrease $P[T=A], P[T=B] \downarrow \delta$

$f(A) + f(B) \geq f(A\cup B) + f(A \cap B)$

min $E[|R|^2]$

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$$\boxed{F^-(p)} \overset{\text{def}}{=} \inf\{E[f(R)] \text{ s.t. } P[e\in R]=p \ \forall e\}$$

Lovász: $F^-(\cancel{p}\,x) = E[f(S_t)]$

w/ $S_t=\{e: \cancel{p_e}\,x_e \ge t\}$, $t\in[0,1]$ unif. random

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Suppose f was not monotone

Lemma $R\subseteq \mathcal{N}$ random, $P[e\in R]\le p \ \forall e$

then $E[f(R)] \ge (1-p)f(\emptyset)$

proof $x_e = P[e\in R]$ $\qquad x \le p\mathbb{1}$

$$E[f(R)] \ge F^-(x) = E[f(S_t)]$$

$$\ge E[f(S_t) \mid t>p]\,P[t>p]$$

$$= f(\emptyset)(1-p)$$

$\mathcal{N}'\subseteq \mathcal{N}$ w/ prob $1/2$

$E[f(\mathcal{N}'\cap S^*)]$

$f(\mathcal{N}'\cap S^*) + f(S^*\setminus \mathcal{N}')$

$\ge f(S^*) + f(\emptyset)$

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   - A. Let $R_i \subseteq \mathcal{N}$ be the k elements e w/ max $f(e \mid S_{i-1})$.
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fix S_{i-1}

$$E[f(e_i|S_{i-1})] = \frac{1}{k} \sum_{e \in R_i} f(e|S_{i-1}) \ge \frac{1}{k} \sum_{d \in S^*} f(d|S_{i-1})$$

$$\ge \dots \ge \frac{1}{k} \left(f(S^* \cup S_{i-1}) - f(S_{i-1}) \right)$$

$$\left(1 - \frac{1}{k}\right)^{k-1} \ge e^{-1}$$

$$E[f(S^* \cup S_{i-1})] \ge \frac{1}{e} f(S^*)$$

$$p \le 1 - \left(1 - \frac{1}{k}\right)^{i-1}$$

$$E[\underbrace{f(e_i|S_{i-1})}_{f(S_i) - f(S_{i-1})}] \ge \frac{1}{k} \left(\left(1 - \frac{1}{k}\right)^{i-1} f(S^*) - E[f(S_{i-1})] \right)$$

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  $f(A) + f(B) \ge f(A \cup B) + f(A \cap B)$

"decreasing marginal values"
for $S \subseteq T \subseteq \mathcal{N}$, $e \in \mathcal{N}$,
$$\underbrace{f(S+e) - f(S)}_{\overset{\text{def}}{=} f(e|S)} \ge \underbrace{f(T+e) - f(T)}_{\overset{\text{def}}{=} f(e|T)}$$

<u>randomized-greedy(f, k)</u>

1. $S_0 \leftarrow \emptyset$
2. For $i = 1, \ldots, k$:
   A. Let $R_i \subseteq \mathcal{N}$ be the k elements e w/ max $f(e \mid S_{i-1})$.
   B. Sample $e_i \sim R$ uniformly at random.
   C. $S_i \leftarrow S_{i-1} + e_i$.
3. Return S_k.

$$q = 1 - \frac{1}{k}$$

$$E[f(S_i)] \ge \frac{1}{k} \cdot \left(1 - \frac{1}{k}\right)^{i-1} f(S^*) + \left(1 - \frac{1}{k}\right) E[f(S_{i-1})]$$

$$E[f(S_k)] \ge \frac{1}{e} OPT$$

$$E[f(S_1)] \ge \frac{1}{k} OPT$$

$$E[f(S_2)] \ge \frac{q}{k} OPT + \frac{q}{k} OPT = \frac{2q}{k}$$

$$E[f(S_3)] \ge \frac{q^2}{k} + \frac{2q^2}{k} = \frac{3q^2}{k}$$

$$E[f(S_i)] \ge \frac{q^{i-1}}{k} OPT + \frac{(i-1)q^{i-1}}{k} OPT = \frac{i q^{i-1}}{k} OPT$$

Lemma $R \subseteq \mathcal{N}$ random, $P[e \in R] \le p$ $\forall e$

then $E[f(R)] \ge (1-p) f(\emptyset)$

Set function $f: 2^{\mathcal{N}} \to \mathbb{R}_{\ge 0}$ s.t.

- Normalized: $f(\emptyset) = 0$
- Nonnegative: $f(S) \ge 0$ $\forall S$
- Monotone: $f(S) \le f(T)$ for $S \subseteq T$
- Submodular: for $A, B \subseteq \mathcal{N}$,

$f(A) + f(B) \ge f(A \cup B) + f(A \cap B)$

"decreasing marginal values"

for $S \subseteq T \subseteq \mathcal{N}$, $e \in \mathcal{N}$,

$\underbrace{f(S+e) - f(S)}_{\overset{\text{def}}{=} f(e|S)} \ge \underbrace{f(T+e) - f(T)}_{\overset{\text{def}}{=} f(e|T)}$

`randomized-greedy(`f`,`k`)`

1. $S_0 \leftarrow \emptyset$
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   A. Let $R_i \subseteq \mathcal{N}$ be the k elements e w/ max $f(e \mid S_{i-1})$.
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