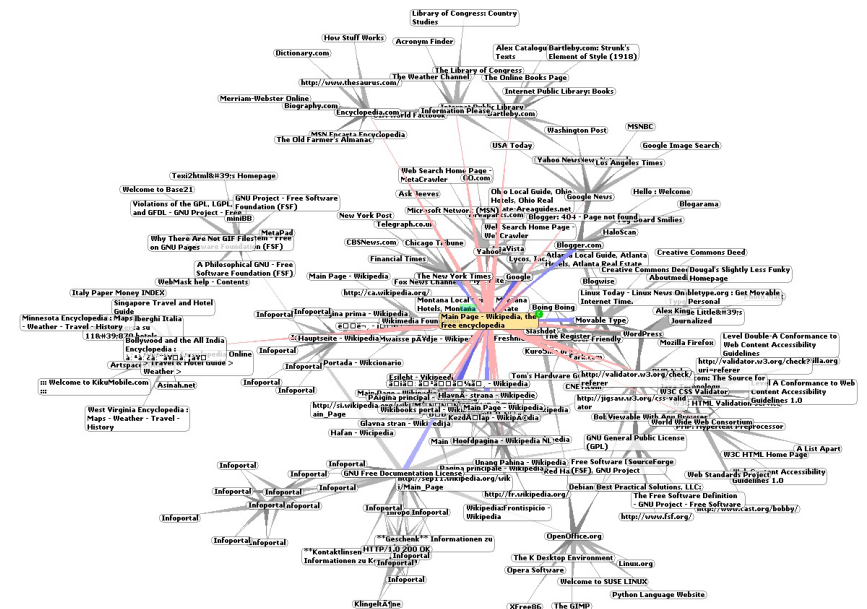


Spectral analysis of undirected random walks

Perron-Frobenius Theorem



Lecture 18

Probability (and techniques)

Algorithms (models and applications)

Hitting times

Cover times

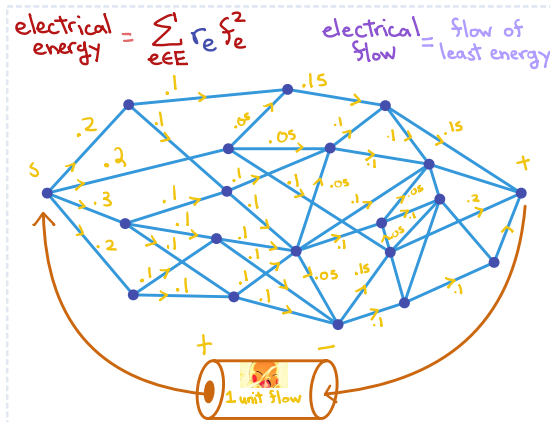
Electrical networks

Connectivity
in $O(\log n)$ space

random walk terminology:

Hitting time $H(s, t)$ = expected # steps until a random walk from s reaches t

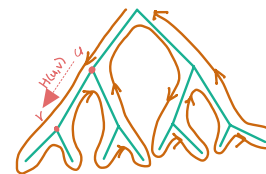
Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex



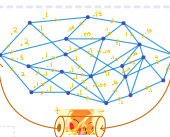
Lemma $H(u, v) + H(v, u) = 2m \cdot (\text{energy of unit flow } (u, v))$

Theorem $C(s) \leq 2mn$

proof Six traversal of spanning tree



$$\begin{aligned} \text{total time} &= \sum_{u, v \in V} H(u, v) + H(v, u) \\ &= 2m \sum_{u, v \in V} (\text{energy of unit flow } (u, v)) \leq 2m(n-1) \end{aligned}$$



electrical energy = $\sum_{e \in E} r_e f_e^2$
electrical flow = least energy

des $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
(BS) $_v$ = net flow at v
 $d_v = \begin{cases} -1 & \text{if } v = s \\ 1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}$

min $\frac{1}{2} \sum_{e \in E} r_e f_e^2$ s.t. $BS = d$
(electrical energy) (is a typical)

effective resistance of d = optimum value/energy

Ohm's law: $S = R^{-1} B^T p$
for electric flow by electric potentials p

$L = BR^{-1}B^T \in \mathbb{R}^{V \times V}$
(effective resist. of d) $\leq \langle d, L^{-1}d \rangle$

Lecture 19

Probability (and techniques)

Algorithms (models and applications)

Spectral theorem
for symmetric maps

Spectral analysis of
random walks

Recall:

$$\text{score}_3(v) = \sum_{(u,v) \in E} \frac{\text{score}_3(u)}{\deg^+(u)}$$



let $\Delta^V = \{x \in \mathbb{R}_V^{\geq 0} : \sum_v x_v = 1\} = \{\text{distributions over } V\}$

Theorem there exists $x \in \Delta^V$ s.t.

$$x_v = \sum_{(u,v) \in E} \frac{x_u}{\deg^+(u)} \quad \forall v \in V.$$

If G is strongly connected, then x is unique and strictly positive.

Theorem, restated: every random walk has a stationary distribution. If G is strongly connected, then this distribution is unique.

Theorem, restated: let G be a dir. graph and R the random walk linear map. Then R has an eigenvector $x \in \Delta^V$ w/ eigenvalue 1. If G is strongly connected, then x is the unique eigenvector w/ eigenvalue 1

Today, for undirected graphs

random walk

Let R have eigenvalues

$$\mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n \geq -1$$

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

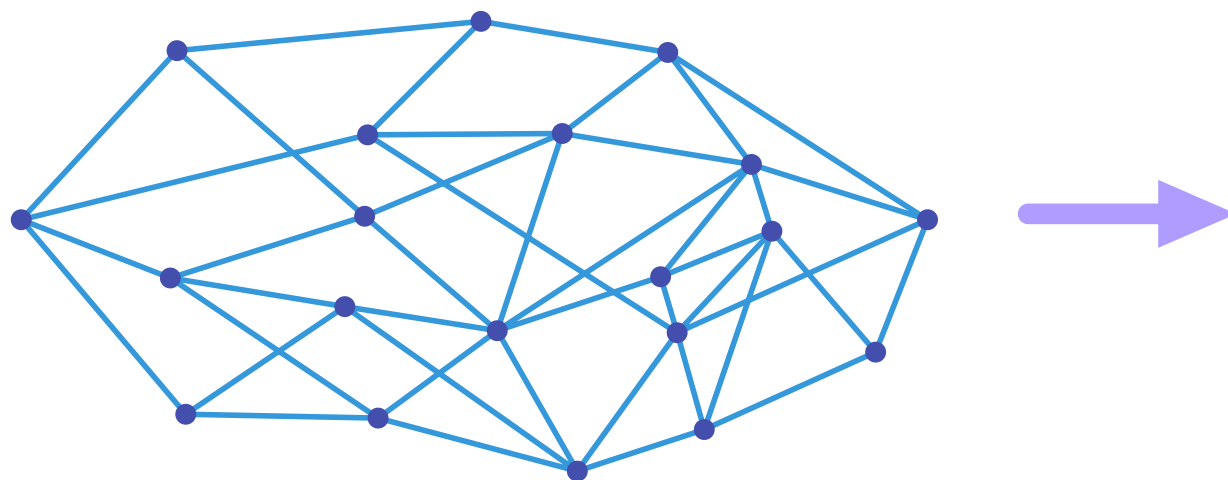
Lecture 19

Probability (and techniques)

Algorithms (models and applications)

Spectral theorem
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Spectral analysis of
random walks



Rank-1 matrix

$$u \otimes v \quad \text{where } u, v \in \mathbb{R}^n$$

(aka. uv^T)

defined by $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

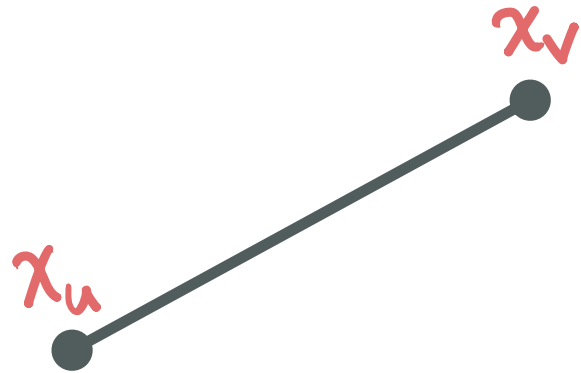
Laplacian of an edge $e = \{u, v\}$

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

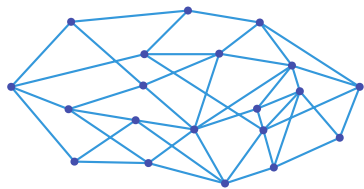
$$L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{V \times V}$$

($1_u = \{0, 1\}$ -indicator vector for u)

$$\langle x, L_e x \rangle = (x_u - x_v)^2$$



Laplacian of



$G=(V,E)$ w/ edge weights $w \in \mathbb{R}_{\geq 0}^E$

$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle =$$

Rank-1 matrix

$u \otimes v$ where $u, v \in \mathbb{R}^n$

$$\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$$

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$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{\substack{e \in E \\ e = \{u, v\}}} w(e) (x_u - x_v)^2$$

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Recall:

$$Ax = \lambda x$$

"eigenvalue" (pointing to λ)

"eigenvector" (pointing to x)

Lemma every linear map A has an eigenvalue $\lambda \in \mathbb{C}$ and eigenvector $x \in \mathbb{C}^n$

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positive semi-definite A :

- A is symmetric
- $\langle x, Ax \rangle \geq 0 \quad \forall x$

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Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,

$x \in \mathbb{R}^n$ maximizes $\langle x, Lx \rangle$ over $\|x\|=1$

$\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

note: $\mu = \langle x, Lx \rangle \in \mathbb{R}$

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proof.

claim: for $u \in \mathbb{R}^n$,

if $\langle u, x \rangle = 0$, then $\langle u, Lx \rangle = 0$.

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claim: if $\langle u, x \rangle = 0$, then $\langle u, Lx \rangle = 0$.

let $u \in \mathbb{R}^n$ w/ $\langle u, x \rangle = 0$, $\|u\|=1$

define $f(\varepsilon) = \frac{\langle x + \varepsilon u, L(x + \varepsilon u) \rangle}{1 + \varepsilon^2}$

(note: $\|x + \varepsilon u\|^2 = \|x\|^2 + \varepsilon^2 \|u\|^2 = 1 + \varepsilon^2$)

x max. $\langle x, Lx \rangle$

$\Rightarrow 0$ max. $f(x)$

$\Rightarrow f'(0) = 0$

$\Rightarrow \langle u, Lx \rangle = 0$

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proof.

✓ for $u \in \mathbb{R}^n$,

if $\langle u, x \rangle = 0$, then $\langle u, Lx \rangle = 0$.

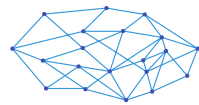
$\Rightarrow Lx \in \text{span}(x)$

$\Rightarrow Lx = \mu_1 x$ for some $\mu_1 \in \mathbb{R}$

$\Rightarrow \mu_1 = \mu_1 \langle x, x \rangle = \langle x, Lx \rangle$

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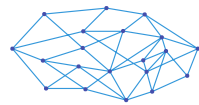
then $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$

where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$

$v_1, \dots, v_n$ orthonormal basis

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$v_1, \dots, v_n$ orthonormal basis

proof by induction on n .

$n=0$ ✓

$n \geq 1$: let $\mu_n \in \mathbb{R}$, $v_n \in \mathbb{R}^n$ be eigenvalue/vector

$$B \stackrel{\text{def}}{=} A - \mu_n(v_n \otimes v_n) \quad Y \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n, \langle v_n, y \rangle = 0\}$$

claim: B maps Y to Y

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where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$

$v_1, \dots, v_n$ orthonormal basis

for $x \in \mathbb{R}^n$, write

$$x = d_1 v_1 + \dots + d_n v_n \quad (\text{unique, } d_i = \langle x, v_i \rangle)$$

$$Lx =$$

$$=$$

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Theorem there exists $x \in \Delta^V$ s.t.

$$x_v = \sum_{(u,v) \in E} \frac{x_u}{\deg^+(u)} \quad \forall v \in V.$$

If G is strongly connected, then x is unique and strictly positive.

Theorem, restated: every random walk has a stationary distribution. If G is strongly connected, then this distribution is unique.

Theorem, restated: let G be a dir. graph and R the random walk linear map. Then R has an eigenvector $x \in \Delta^V$ w/ eigenvalue 1. If G is strongly connected, then x is the unique eigenvector w/ eigenvalue 1

Theorem: $G=(V,E)$ undirected, $w:E \rightarrow \mathbb{R}_{>0}$

let $R:\mathbb{R}^V \rightarrow \mathbb{R}^V$ be random walk map

then R has stationary distr. $s \in \Delta_V$ w/

$$s_v = \frac{\deg(v)}{2W}$$

$$\deg(v) = \sum_{e \in \partial(v)} w(e)$$

$$W = \sum_{e \in E} w(e)$$

Proof

Rank-1 matrix
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Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

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Theorem: $G=(V,E)$ undirected, $w:E \rightarrow \mathbb{R}_{>0}$
 then R has stationary distr. $s \in \Delta_V$ w/

$$s_v = \frac{\deg(v)}{2W} \quad \deg(v) = \sum_{e \in \mathcal{E}(v)} w(e), \quad W = \sum_{e \in E} w(e)$$

Lemma let $A = C^{-1}BC$.

- A and B have same eigenvalues
- C maps eigenvectors of A to eigenvectors of B

Normalized random walk matrix

$$Q = D^{-1/2} R D^{1/2} = D^{-1/2} A D^{1/2}$$

- $D = \text{diag}(d)$ w/ $d = \text{degrees}$
- $A = \text{adjacency matrix}$
- $R = A D^{-1} = \text{random walk matrix}$
- Q is symmetric!
- Q has same eigenvalues as R !

Similar matrices

$A, B \in \mathbb{R}^{n \times n}$ are similar if

$$A = C^{-1}BC \text{ for some invertible } C \in \mathbb{R}^{n \times n}$$

let $A = C^{-1}BC$.

Lemma C restricts to isomorphism
between $\ker(A)$ and $\ker(B)$.

proof:

Rank-1 matrix
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$$s_v = \frac{\deg(v)}{2W} \quad \deg(v) = \sum_{e \in \mathcal{E}(v)} w(e), \quad W = \sum_{e \in E} w(e)$$

let $A = C^{-1}BC$.

Lemma C restricts to isomorphism
between $\ker(A)$ and $\ker(B)$.

Lemma

- A and B have same eigenvalues
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why?

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$$Q = v_1 \otimes v_1 + \mu_2 (v_2 \otimes v_2) + \dots + \mu_n (v_n \otimes v_n)$$

w/ $\mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n$
(also eigenvalues of R)

$$D^{1/2} v_1 \xleftrightarrow{\text{proportional}} \text{stationary distribution} \xleftrightarrow{\text{proportional}} d$$

$$\Rightarrow v_1 = \frac{\sqrt{d}}{\|\sqrt{d}\|} = \frac{\sqrt{d}}{\sqrt{\langle d, \mathbf{1} \rangle}}$$

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