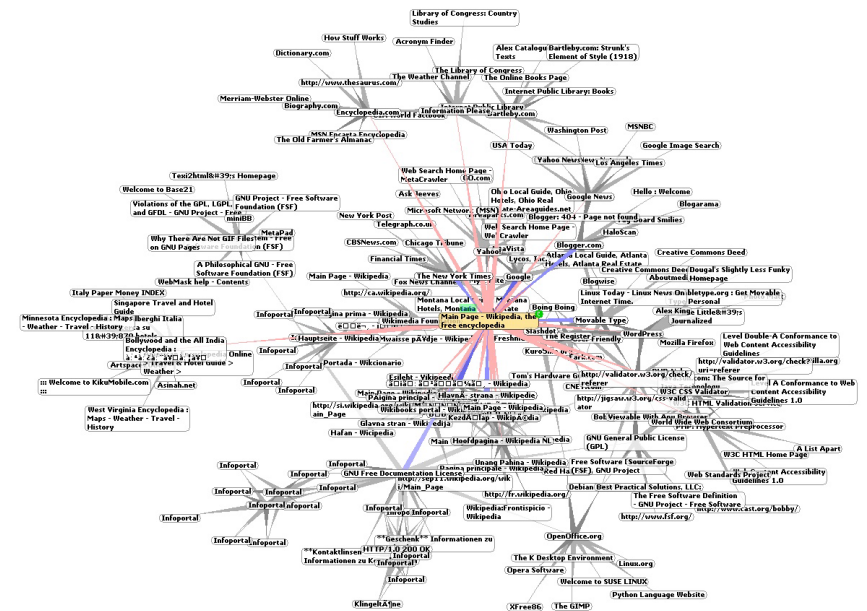


Spectral analysis of undirected random walks

Perron-Frobenius Theorem



Lecture 18

Probability (and techniques)

Algorithms (models and applications)

Hitting times

Cover times

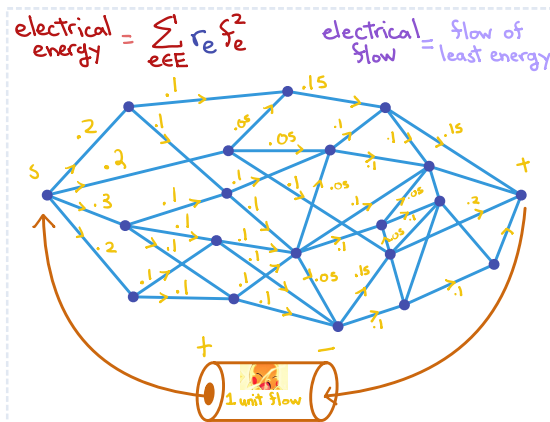
Electrical networks

Connectivity
in $O(\log n)$ space

random walk terminology:

Hitting time $H(s, t)$ = expected # steps until a random walk from s reaches t

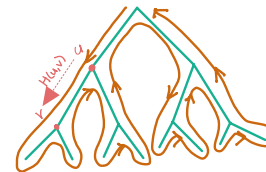
Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex



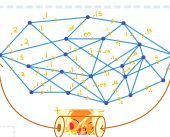
Lemma $H(u, v) + H(v, u) = 2m \cdot (\text{energy of unit flow } (u, v))$

Theorem $C(s) \leq 2mn$

proof Six traversal of spanning tree



$$\begin{aligned} \text{total time} &= \sum_{u, v \in V} H(u, v) + H(v, u) \\ &= 2m \sum_{u, v \in V} (\text{energy of unit flow } (u, v)) \leq 2m(n-1) \end{aligned}$$



electrical energy = $\sum_{e \in E} r_e f_e^2$
electrical flow = least energy

des $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
(BS) $_v$ = net flow at v
 $d_v = \begin{cases} -1 & \text{if } v = s \\ 1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}$

min $\frac{1}{2} \sum_{e \in E} r_e f_e^2$ s.t. $BS = d$
(electrical energy) (is a typical)

effective resistance of d = optimum value/energy
Ohm's law: $f = R^{-1} B^T p$
for electric flow by electric potentials p
 $L = BR^{-1}B^T \in \mathbb{R}^{V \times V}$
(effective resist. of d) $\leq \langle d, L^{-1}d \rangle$

Lecture 19

Probability (and techniques)

Algorithms (models and applications)

Spectral theorem
for symmetric maps

Spectral analysis of
random walks

Recall:

$$\text{score}_3(v) = \sum_{(u,v) \in E} \frac{\text{score}_3(u)}{\deg^+(u)}$$



let $\Delta^V = \{x \in \mathbb{R}_V^{\geq 0} : \sum_v x_v = 1\} = \{\text{distributions over } V\}$

Theorem there exists $x \in \Delta^V$ s.t.

$$x_v = \sum_{(u,v) \in E} \frac{x_u}{\deg^+(u)} \quad \forall v \in V.$$

If G is strongly connected, then x is unique and strictly positive.

Theorem, restated: every random walk has a stationary distribution. If G is strongly connected, then this distribution is unique.

Theorem, restated: let G be a dir. graph and R the random walk linear map. Then R has an eigenvector $x \in \Delta^V$ w/ eigenvalue 1. If G is strongly connected, then x is the unique eigenvector w/ eigenvalue 1

Today, for undirected graphs

random walk

Let R have eigenvalues

$$\mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n \geq -1$$

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

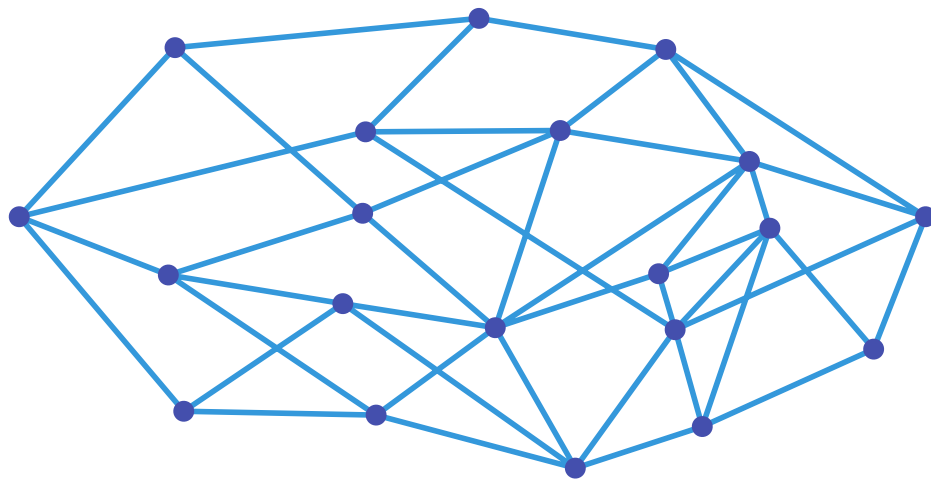
Lecture 19

Probability (and techniques)

Algorithms (models and applications)

Spectral theorem
for symmetric maps

Spectral analysis of
random walks



Laplacian

$$L \in \mathbb{R}^{n \times n}$$

(symmetric, positive semidefinite)

Rank-1 matrix

$$u \otimes v \quad \text{where } u, v \in \mathbb{R}^n$$

(aka. uv^T)

defined by $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

as a matrix:

$$u \otimes v = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_n v_1 & u_n v_2 & & u_n v_n \end{bmatrix}$$

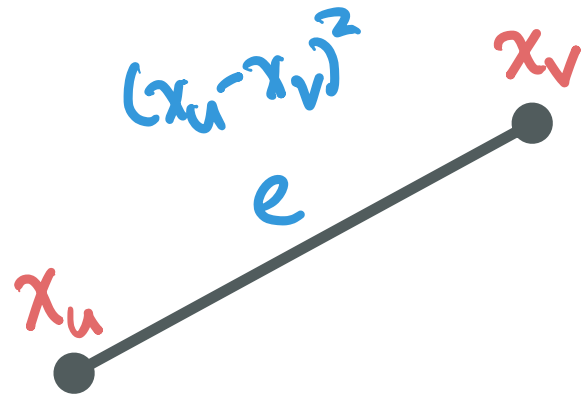
Laplacian of an edge $e = \{u, v\}$

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

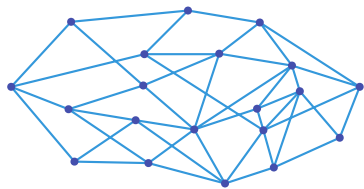
$$L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{V \times V}$$

($1_u = \{0, 1\}$ -indicator vector for u)

$$\langle x, L_e x \rangle = (x_u - x_v)^2$$



Laplacian of



$G=(V,E)$ w/ edge weights $w \in \mathbb{R}_{\geq 0}^E$

$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{\substack{e \in E \\ e=\{u,v\}}} w(e) (x_u - x_v)^2$$

Rank-1 matrix

$u \otimes v$ where $u, v \in \mathbb{R}^n$

$$\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$$

Laplacian of $e = \{u, v\}$

$$L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{V \times V}$$

($1_u = \{0,1\}$ -indicator vector for u)

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$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{\substack{e \in E \\ e = \{u, v\}}} w(e) (x_u - x_v)^2$$

Rank-1 matrix
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Laplacian of $e = \{u, v\}$

$$L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{n \times n}$$

($1_u = \{0, 1\}$ -indicator vector for u)

$$\langle x, L_e x \rangle = (x_u - x_v)^2$$

$A \in \mathbb{R}^{n \times n}$ is positive semi-definite if:

- A is symmetric
- $\langle x, Ax \rangle \geq 0 \quad \forall x$

$$\begin{aligned} \langle x, (y \otimes y) x \rangle \\ = \langle x, y \rangle^2 \geq 0 \end{aligned}$$

(A is positive definite if it is positive semi-definite & invertible)

Recall:

$$Ax = \lambda x$$

"eigenvalue" (pointing to λ)

"eigenvector" (pointing to x)

Lemma every linear map A has an eigenvalue $\lambda \in \mathbb{C}$ and eigenvector $x \in \mathbb{C}^n$

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Laplacian of $e = \{u, v\}$

$$L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{n \times n}$$

($1_u = \{0, 1\}$ -indicator vector for u)

$$\langle x, L_e x \rangle = (x_u - x_v)^2$$

Laplacian of



$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{e = \{u, v\}} w(e) (x_u - x_v)^2$$

positive semi-definite A :

- A is symmetric
- $\langle x, Ax \rangle \geq 0 \quad \forall x$

Lemma every linear map A has an eigenvalue $\lambda \in \mathbb{C}$ and eigenvector $x \in \mathbb{C}^n$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,

$x \in \mathbb{R}^n$ maximizes $\langle x, Lx \rangle$ over $\|x\|=1$

$\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

note: $\mu = \langle x, Lx \rangle \in \mathbb{R}$

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Laplacian of $e = \{u, v\}$

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($1_u = \{0, 1\}$ -indicator vector for u)

$$\langle x, L_e x \rangle = (x_u - x_v)^2$$

Laplacian of 

$$L = \sum_{e \in E} w(e) L_e$$

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positive semi-definite A :

- A is symmetric
- $\langle x, Ax \rangle \geq 0 \quad \forall x$

"eigenvalue"

$$Ax = \lambda x$$

"eigenvector"

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric, $x \neq 0$
 $x \in \mathbb{R}^n$ maximizes $\frac{\langle x, Lx \rangle}{\langle x, x \rangle}$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

proof:

claim: for $u \in \mathbb{R}^n$, $x \rightarrow x + \epsilon u$
 $\|x\|=1 \Rightarrow \|x\|^2 + \epsilon^2 \|u\|^2 = 1 + \epsilon^2$

if $\langle u, x \rangle = 0$, then $\langle u, Lx \rangle = 0$.

if claim is true: $\langle x + \epsilon u, L(x + \epsilon u) \rangle$
 $= \langle x, Lx \rangle + 2\epsilon \langle u, Lx \rangle + \epsilon^2 \langle u, Lu \rangle$
 $\Rightarrow Lx \in \text{span}(x)$

$\Rightarrow Lx = \mu_1 x$ for some $\mu_1 \in \mathbb{R}$

$\Rightarrow \mu_1 = \mu_1 \langle x, x \rangle = \langle x, Lx \rangle$

Rank-1 matrix
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 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Laplacian of $e = \{u, v\}$
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- $\langle x, Ax \rangle \geq 0 \quad \forall x$

"eigenvalue"
 $Ax = \lambda x$
 "eigenvector"

$\frac{\langle x + \epsilon u, L(x + \epsilon u) \rangle}{\langle x + \epsilon u, x + \epsilon u \rangle}$

$L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ maximizes $\langle x, Lx \rangle$ over $\|x\|=1$

claim: if $\langle u, x \rangle = 0$, then $\langle u, Lx \rangle = 0$.

let $u \in \mathbb{R}^n$ w/ $\langle u, x \rangle = 0$, $\|u\|=1$

define $f(\varepsilon) = \frac{\langle x + \varepsilon u, L(x + \varepsilon u) \rangle}{1 + \varepsilon^2}$

(note: $\|x + \varepsilon u\|^2 = \|x\|^2 + \varepsilon^2 \|u\|^2 = 1 + \varepsilon^2$)

Rank-1 matrix
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 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

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Laplacian of 
 $L = \sum_{e \in E} w(e) L_e$
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let $u \in \mathbb{R}^n$ w/ $\langle u, x \rangle = 0$, $\|u\|=1$

define $f(\varepsilon) = \frac{\langle x + \varepsilon u, L(x + \varepsilon u) \rangle}{1 + \varepsilon^2}$

(note: $\|x + \varepsilon u\|^2 = \|x\|^2 + \varepsilon^2 \|u\|^2 = 1 + \varepsilon^2$)

x max. $\langle x, Lx \rangle$

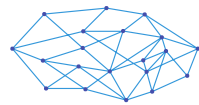
$\Rightarrow 0$ max. $f(x)$

$\Rightarrow f'(0) = 0$

$\Rightarrow \langle u, Lx \rangle = 0$

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 $x \in \mathbb{R}^n$ maximizes $\langle x, Lx \rangle$ over $\|x\|=1$
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proof.

✓ for $u \in \mathbb{R}^n$,

✓ if $\langle u, x \rangle = 0$, then $\langle u, Lx \rangle = 0$.

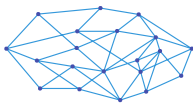
$$\Rightarrow Lx \in \text{span}(x)$$

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 $x \in \mathbb{R}^n$ maximizes $\langle x, Lx \rangle$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric

then $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$

where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$

$v_1, \dots, v_n$ orthonormal basis

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Laplacian of $e = \{u, v\}$
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Laplacian of 
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$$\text{then } L = \lambda_1(v_1 \otimes v_1) + \dots + \lambda_n(v_n \otimes v_n)$$

where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$

$v_1, \dots, v_n$ orthonormal basis

proof by induction on n .

$n=0$ ✓

$n \geq 1$: let $\mu_n \in \mathbb{R}$, $v_n \in \mathbb{R}^n$ be eigenvalue/vector

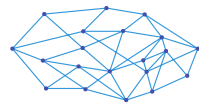
$$B \stackrel{\text{def}}{=} A - \mu_n(v_n \otimes v_n) \quad Y \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n, \langle v_n, y \rangle = 0\}$$

claim: B maps Y to Y $\langle v_n, By \rangle = 0 \quad \forall y \in Y$

$$\langle v_n, By \rangle = \langle v_n, (A - \mu_n(v_n \otimes v_n))y \rangle$$

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v)y \rangle = \langle x, u \rangle \langle v, y \rangle$

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Laplacian of 
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positive semi-definite A :

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- $\langle x, Ax \rangle \geq 0 \quad \forall x$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ maximizes $\langle x, Lx \rangle$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

$$= \langle v_n, A \gamma \rangle - \mu_n \langle v_n, (v_n \otimes v_n) \gamma \rangle$$

$$= \langle A v_n, \gamma \rangle - \mu_n \underbrace{\langle v_n, v_n \rangle}_{=1} \underbrace{\langle v_n, \gamma \rangle}_0$$

$$= \mu \langle v_n, \gamma \rangle = 0$$

$$B \rightarrow \mu_1, \dots, \mu_{n-1}$$

$$v_1, \dots, v_{n-1} \in Y$$

Rank-1 matrix

$u \otimes v$ where $u, v \in \mathbb{R}^n$

$$\langle x, (u \otimes v) \gamma \rangle = \langle x, u \rangle \langle v, \gamma \rangle$$

Laplacian of $e = \{u, v\}$

$$L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{V \times V}$$

($1_u = \{0,1\}$ -indicator vector for u)

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Laplacian of



$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, L x \rangle = \sum_{e = \{u, v\}} w(e) (x_u - x_v)^2$$

positive semi-definite A :

- A is symmetric
- $\langle x, A x \rangle \geq 0 \quad \forall x$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,

$x \in \mathbb{R}^n$ maximizes $\langle x, L x \rangle$ over $\|x\|=1$

$$\Rightarrow L x = \mu x \quad \text{for } \mu = \langle x, L x \rangle$$

Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric

$$L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$$

where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$

$v_1, \dots, v_n$ orthonormal basis

For $x \in \mathbb{R}^n$, write

$$x = d_1 v_1 + \dots + d_n v_n \quad (\text{unique, } d_i = \langle x, v_i \rangle)$$

$$Lx = \lambda_1 (v_1 \otimes v_1)x + \dots + \lambda_n (v_n \otimes v_n)x$$

$$= \lambda_1 \langle v_1, x \rangle v_1 + \dots + \lambda_n \langle v_n, x \rangle v_n$$

$$= d_1 \lambda_1 v_1 + \dots + d_n \lambda_n v_n$$

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v)y \rangle = \langle x, u \rangle \langle v, y \rangle$

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Laplacian of 

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- A is symmetric
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 $x \in \mathbb{R}^n$ maximizes $\langle x, Lx \rangle$ over $\|x\|=1$
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Lecture 19

Probability (and techniques)

Algorithms (models and applications)

✓ Spectral theorem
for symmetric maps

Spectral analysis of
random walks

Recall:

$$\text{score}_3(v) = \sum_{(u,v) \in E} \frac{\text{score}_3(u)}{\deg^+(u)}$$



let $\Delta^V = \{x \in \mathbb{R}_V^{\geq 0} : \sum_v x_v = 1\} = \{\text{distributions over } V\}$

Theorem there exists $x \in \Delta^V$ s.t.

$$x_v = \sum_{(u,v) \in E} \frac{x_u}{\deg^+(u)} \quad \forall v \in V.$$

If G is strongly connected, then x is unique and strictly positive.

Theorem, restated: every random walk has a stationary distribution. If G is strongly connected, then this distribution is unique.

Theorem, restated: let G be a dir. graph and R the random walk linear map. Then R has an eigenvector $x \in \Delta^V$ w/ eigenvalue 1. If G is strongly connected, then x is the unique eigenvector w/ eigenvalue 1

Theorem: $G=(V,E)$ undirected, $w:E \rightarrow \mathbb{R}_{>0}$

let $R:\mathbb{R}^V \rightarrow \mathbb{R}^V$ be random walk map

then R has stationary distr. $s \in \Delta_V$ w/

$$s_v = \frac{\deg(v)}{2W}$$

$$\deg(v) = \sum_{e \in \mathcal{E}(v)} w(e)$$

$$W = \sum_{e \in E} w(e)$$

Proof

$$s = \frac{A\mathbf{1}}{2W}$$

$$R = AD^{-1}$$

$$R = A \frac{D^{-1}d}{\langle \mathbf{1}, d \rangle}$$

$$= \frac{A\mathbf{1}}{\langle \mathbf{1}, d \rangle} = \frac{d}{\langle \mathbf{1}, d \rangle}$$

$$d = A\mathbf{1}$$

$$D = \begin{bmatrix} d & & \\ & \ddots & \\ & & d \end{bmatrix}$$

$$s = \frac{d}{\langle \mathbf{1}, d \rangle}$$

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Laplacian of $e = \{u, v\}$

$$L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{V \times V}$$

($1_u = \{0,1\}$ -indicator vector for u)

$$\langle x, L_e x \rangle = (x_u - x_v)^2$$

Laplacian of



$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

positive semi-definite A :

- A is symmetric
- $\langle x, Ax \rangle \geq 0 \quad \forall x$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
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 $v_1, \dots, v_n$ orthonormal basis

Let R have eigenvalues

$$\mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n \geq -1$$

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

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Similar matrices

$A, B \in \mathbb{R}^{n \times n}$ are similar if

$$A = C^{-1}BC \text{ for some invertible } C \in \mathbb{R}^{n \times n}$$

let $A = C^{-1}BC$.

Lemma C restricts to isomorphism
between $\ker(A)$ and $\ker(B)$.

$$x \in \ker(A)$$

$$BCx = CAx = 0$$

$$y \in \ker(B)$$

$$AC^{-1}y = C^{-1}By = 0$$

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Lemma

- A and B have same eigenvalues
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why?

$$A - \lambda I = C^{-1}(B - \lambda I)C \quad \forall \lambda$$

$$C^{-1}(\lambda I)C = \lambda I$$

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$$D^{1/2} v_1 \xleftrightarrow{\text{proportional}} \text{stationary distribution} \xleftrightarrow{\text{proportional}} d$$

$$\Rightarrow v_1 = \frac{\sqrt{d}}{\|\sqrt{d}\|} = \frac{\sqrt{d}}{\sqrt{\langle d, \mathbf{1} \rangle}}$$

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$$R^k x = (D^{1/2} Q D^{-1/2})^k x \rightsquigarrow S$$

$$= (D^{1/2} Q D^{-1/2}) (D^{1/2} Q D^{-1/2}) \dots (D^{1/2} Q D^{-1/2}) x$$

$$= D^{1/2} Q^k D^{-1/2} x$$

$$= D^{1/2} \left(\frac{\sqrt{d} \otimes \sqrt{d}}{\langle \mathbf{1}, d \rangle} + \mu_2 (v_2 \otimes v_2) + \dots + \mu_n (v_n \otimes v_n) \right)^k D^{-1/2} x$$

$$= D^{1/2} \left(\frac{\sqrt{d} \otimes \sqrt{d}}{\langle \mathbf{1}, d \rangle} + \mu_2^k (v_2 \otimes v_2) + \dots + \mu_n^k (v_n \otimes v_n) \right) D^{-1/2} x$$

$$D^{1/2} \frac{\sqrt{d} \otimes \sqrt{d}}{\langle \mathbf{1}, d \rangle} D^{-1/2} x = \frac{d \otimes \mathbf{1}}{\langle \mathbf{1}, d \rangle} x = \frac{\langle \mathbf{1}, x \rangle d}{\langle \mathbf{1}, d \rangle}$$

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$$= \langle D^{1/2} W D^{-1/2} x, D^{1/2} W D^{-1/2} x \rangle$$

$$= \langle W D^{-1/2} x, D (W D^{-1/2} x) \rangle$$

$$\leq \Delta_{\max} \langle W D^{-1/2} x, W D^{-1/2} x \rangle$$

$$= \Delta_{\max} \langle D^{-1/2} x, W^2 D^{-1/2} x \rangle$$

$$\leq \Delta_{\max} \max \{ \mu_2, \mu_n \} \langle D^{-1/2} x, D^{-1/2} x \rangle$$

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$$\leq \frac{\Delta_{\max}}{\Delta_{\min}} \max \{ \mu_2, \mu_n \} \|x\|^2 \leq 1$$

$$(1-\delta)^k$$

$$\delta = 1 - \max \{ \mu_2, |\mu_n| \}$$

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