

Deterministic connectivity
in logarithmic space

Lecture 1

Probability

Events, probabilities,
conditional probabilities,
union bound

Random variables,
expected value,
linearity of expectation
Markov's inequality

Algorithms

Approx. 3-SAT,

Big idea: we only want the average

$$\begin{aligned} \text{Linearity of Expectation} \\ E[X+Y] &= E[X] + E[Y] \\ \text{"average sum = sum of averages"} \end{aligned}$$

let $Y_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{otherwise} \end{cases}$ ($i=1, \dots, m$)

$$E[Y_i] = 7/8 \quad (1 \text{ clause case})$$

$$E[\text{\# clauses satisfied}] = E\left[\sum_{i=1}^m Y_i\right] = \sum_{i=1}^m E[Y_i] = 7/8 m$$

let $Y_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{otherwise} \end{cases}$

$$Z = \sum_{i=1}^m Y_i = \text{\# clauses satisfied}$$

$$E[Z] = \frac{1}{2} E[Z | x_1 = \text{true}] + \frac{1}{2} E[Z | x_1 = \text{false}]$$

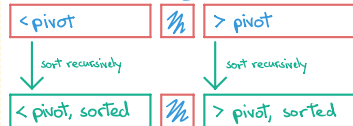
$\Rightarrow$ either $E[Z | x_1 = \text{true}]$ or $E[Z | x_1 = \text{false}]$ is $\geq E[Z]$!

QuickSort,

Random divide and conquer

① pick random 'pivot'

② Divide A into two parts



Expected running time of QuickSort

For each $i, j \in [n]$ w/ $i < j$,

$X_{ij} = \begin{cases} 1 & \text{if } i\text{th smallest element is compared to } j\text{th smallest} \\ 0 & \text{otherwise} \end{cases}$

$\sum_{i,j} X_{ij}$ = total # comparisons made by the algorithm
 $\approx$ total running time

$E[\sum_{i,j} X_{ij}]$ = average running time

$E[\sum_{i,j} X_{ij}] = \sum_{i,j} E[X_{ij}]$

now analyze $E[X_{ij}]$ in isolation

$$E[X_{ij}] = \Pr[X_{ij} = 1] = \Pr[i\text{th element is compared to } j\text{th}]$$

i th and j th elem selected before any element in between happens w/ probability $\frac{2}{j-i+1}$

$$E[X_{ij}] = \frac{2}{j-i+1}$$

by linearity of expectation

$$E[\sum_{i,j} X_{ij}] = \sum_{i,j} E[X_{ij}] = \sum_{i,j} \frac{2}{j-i+1}$$

$$\leq 2n \left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right)$$

$$\leq O(n \log n) \quad \text{with harmonic series}$$

Quickselect (HW)

Lecture 2

Probability

Hash Functions,

Universal Hash Functions

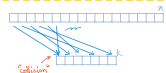
$h: [n] \rightarrow k$ s.t. for all pairs $i, j \in [n]$,
 $\Pr[h(i) = h(j)] \leq \frac{1}{k}$ (much weaker than ideal hash)

e.g. $h(x) = (ax + b) \bmod p \bmod k$

where • p prime, $> k$

• a in $\{1, \dots, p-1\}$ unif. random

• b in $\{0, \dots, p-1\}$ unif. random



Markov's Inequality

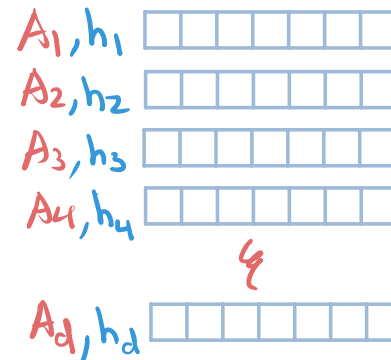
Markov's Inequality (special case)

Let $X \geq 0$ be nonnegative RV

$$\Pr[X \geq 2E[X]] \leq \frac{1}{2}$$

Algorithms

Frequency Estimation, Heavy Hitters (in streams)



estimate f_i w/
 $\min_{j=1, \dots, d} A_j[h_j(i)]$

Lecture 3

Probability

k-wise independence,
higher order moments
(generalizing universal hash
functions and Markov's inequality)

Algorithms

Hash tables w/
chaining +
linear probing

k-wise independent hash functions

$$h: [U] \rightarrow [m]$$

s.t. for any k keys $x_1, x_2, \dots, x_k \in [U]$,
 k values $y_1, y_2, \dots, y_k \in [m]$,

$$P[h(x_1)=y_1, h(x_2)=y_2, \dots, h(x_k)=y_k] = \frac{1}{m^k}$$

eg. $h(x) = a_0 + a_1x + a_2x^2 + \dots + a_{k-1}x^{k-1} \pmod{p}$

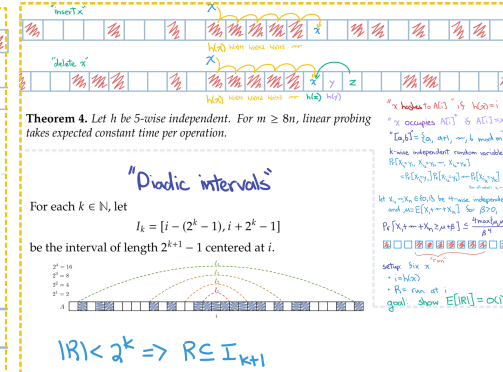
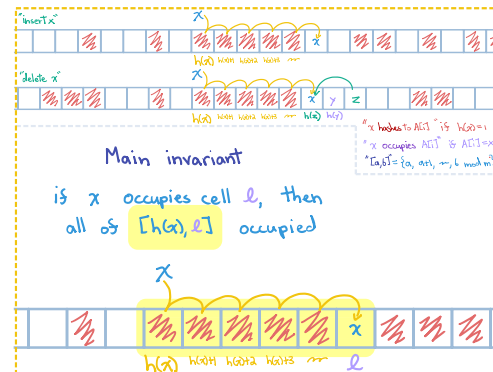
where $p \geq U$ prime,
 $a_0, a_1, \dots, a_{k-1} \in [p]$ indep, uniformly random

Important lemma

k-wise independent random variables
 $P[X_1=y_1, X_2=y_2, \dots, X_k=y_k]$
 $= P[X_1=y_1]P[X_2=y_2] \dots P[X_k=y_k]$
for all values $y_1, \dots, y_k$

let $X_1, \dots, X_n \in \{0, 1\}$ be 4-wise independent
and $\mu = E[X_1 + \dots + X_n]$. then for $\beta > 0$,

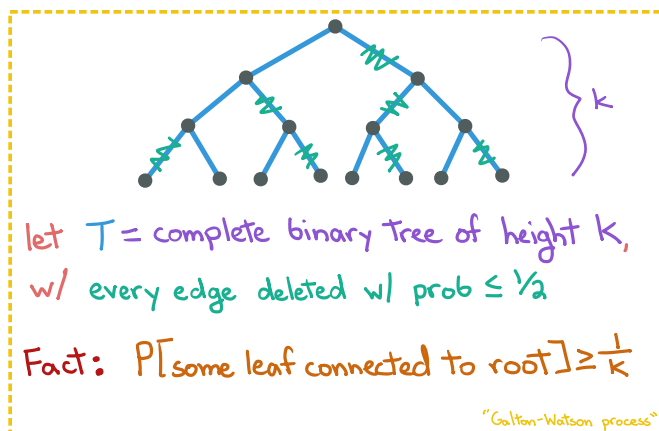
$$P[X_1 + \dots + X_n \geq \mu + \beta] \leq \frac{4 \max\{\mu, \mu^3\}}{\beta^4}$$



Lecture 4

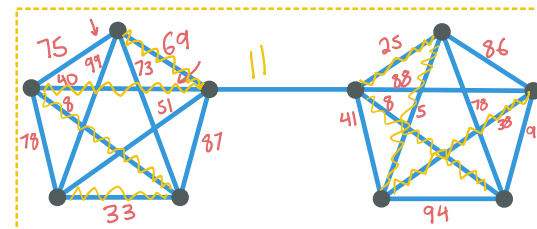
Probability

Galton-Watson processes



Algorithms

Random Contractions for min cut



Randomized contractions

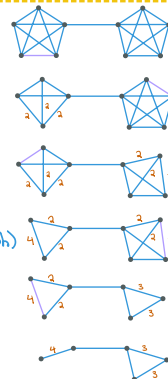
Input: $G=(V,E)$, weights $c \in \mathbb{R}_{\geq 0}^E$ (unweighted)

① while $|E| > 1$

② sample $e \in E$ in proportion to c

③ "contract" e

④ return set of edges (in original graph) contracted to remaining edge



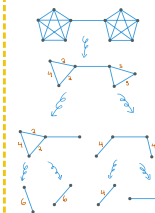
Amplification by branching

Instead of restarting at the beginning, restart partway through

High-level algorithm

① randomly contract until prob. of ruining the min-cut is $\sim 1/2$

② Branch:
make two copies of current graph
recurse on both copies



Lecture 5

Probability

Multiplicative Chernoff Bound

Multiplicative Chernoff bound

let $X_1, \dots, X_n \in [0, 1]$ be independent

$$\mu = E[X_1 + \dots + X_n]$$

$$\epsilon \in (0, 1)$$

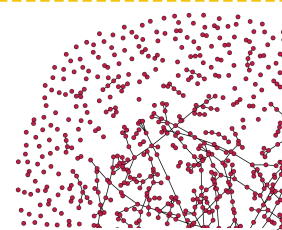
- $P[X_1 + \dots + X_n \geq (1 + \epsilon)\mu] \leq e^{-\epsilon^2 \mu / 3}$
- $P[X_1 + \dots + X_n \leq (1 - \epsilon)\mu] \leq e^{-\epsilon^2 \mu / 2}$

Algorithms (/ results)

Random graphs & the giant component

Erdős-Rényi graphs $G(n, p)$

- n vertices
- each edge sampled independently w/ probability p



Theorem 5.3. Consider a random graph $G \sim \mathcal{G}(n, p)$ for $p = c/n$, where c is a constant.

- If $c > 1$, then with high probability, there is exactly one connected component of G with $\Omega(n)$ vertices, and all other components have size $\leq O(\log n)$.
- For $c < 1$, then with high probability, all connected components of G has size $< O(\log n)$.

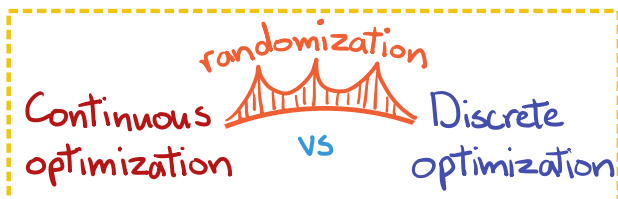
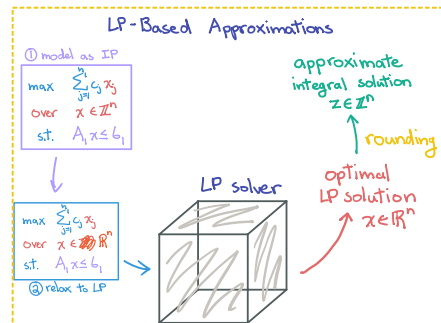
Lecture 6

Probability (& techniques)

Approximations via linear programming & randomized rounding

Linear programs (LPs)
Similar to IPs, except no integrality constraints

min/max $\sum_{j=1}^n c_j x_j$ linear objective
over $x \in \mathbb{R}^n$ linear integral variables
st. $A_1 x \leq b_1$ linear constraints
 $A_2 x = b_2$
 $A_3 x \geq b_3$



Algorithms

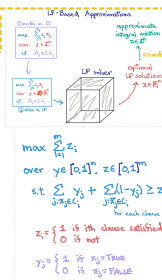
Max Sat Set Cover Covering Integer Programs

For each variable x_j , independently, randomly set $x_j = t$ with probability g_j , and $x_j = f$ otherwise.

$$E[\# \text{ sat. clauses}] \geq (1 - 1/e) \text{OPT}$$

k	oblivious	LP	average
1	1/2	$z_i \geq (3/4)z_i$	$\geq (3/4)z_i$
2	3/4	$(3/4)z_i \geq (3/4)z_i$	$\geq (3/4)z_i$
3	7/8	$(1 - 1/e)z_i \geq (3/4)z_i$	$\geq (3/4)z_i$
4	15/16	$(1 - 1/e)z_i \geq (3/4)z_i$	$\geq (3/4)z_i$
5	31/32	$(1 - 1/e)z_i \geq (3/4)z_i$	$\geq (3/4)z_i$
...	...	...	...
k	$1 - 1/2^k$	$(1 - 1/e)z_i \geq (3/4)z_i$	$\geq (3/4)z_i$

Can we get $\frac{3}{4}$ -APX?



Rounding LP solution
 $x \in [0, 1]^n \rightsquigarrow F \subseteq \{S_1, \dots, S_n\}$

For each set S_j , independently, take S_j with probability x_j .

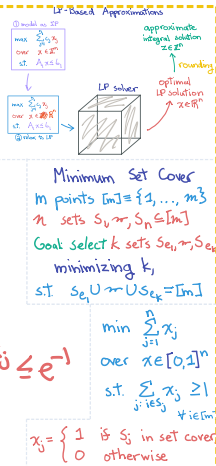
$$E[|F|] = \sum x_j \leq \text{OPT}$$

Fix a point i .

$$E[\# \text{ sets in } F \text{ covering } i] = \sum_{j \in S_i} x_j \geq z_i$$

$$P[F \text{ not cover } i] = \prod_{j \in S_i} (1 - x_j) \leq e^{-\sum_{j \in S_i} x_j} \leq e^{-z_i}$$

$$P[F \text{ cover } i] \geq 1 - e^{-1}$$



Lecture 7

Probability

Continuous Random Variables

Variance, Chebyshev's Variance Reduction

Median trick

Variance

$$\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

Chebyshev's inequality

$$P[|X - E[X]| \geq d] \leq \frac{\text{Var}(X)}{d^2}$$

"median trick", "median of means"

- ① take mean to reduce variance, get constant prob. of error
- ② take median to amplify success prob. exponentially

Algorithms

Distinct Elements



Goal: estimate # of distinct elements in the stream

Suppose you had an ideal hash function

$$h: [m] \rightarrow [0, 1]$$

and the following facts:

let $Y_1, \dots, Y_k \in [0, 1]$ independent and uniformly random

$$X = \min \{Y_1, \dots, Y_k\}$$

$$\text{then } E[X] = \frac{1}{k+1}$$

$$\textcircled{1} X \leftarrow 1$$

② for each element e

$$\textcircled{a} X \leftarrow \min\{X, h(e)\}$$

$$/* E[X] = \frac{1}{k+1}, k = (\text{\# distinct elements}) */$$

$$\textcircled{3} \text{ return } \frac{1}{X} - 1$$

$$\text{recall } E[X] = \frac{1}{k+1}$$

$$\text{goal: show } |X - E[X]| \leq \frac{\epsilon}{k+1}$$

w/ good prob.

$$P[|X - E[X]| \geq \frac{\epsilon}{k+1}]$$

$$\leq \text{Var}(X) \cdot \left(\frac{k+1}{\epsilon}\right)^2 \leftarrow \text{Chebyshev's inequality}$$

where

$$\text{Var}(X) = \frac{k}{(k+1)^2(k+2)} \approx \frac{1}{k^2}$$

Q: how to reduce $\text{Var}(X)$?

we can get $(1 \pm \epsilon)$ -APX

w/ 10% error prob. in $O(\frac{1}{\epsilon^2})$ space

what about high probability?

① make q copies $Y_1, \dots, Y_q$ of

② return median of $Y_1, \dots, Y_q$

Algo correct $\Leftrightarrow \geq \frac{1}{2}$ of Y_i 's correct

let $Z = \# \text{correct } Y_i$'s $E[Z] = \frac{q}{2}$

$$P[Z \leq (1 - \frac{\epsilon}{4})E[Z]] \leq e^{-\left(\frac{\epsilon}{4}\right)^2 E[Z]/2} \leq \frac{1}{n^{O(n)}} \text{ for } q = O(\log n)$$

ideal hash $h: [m] \rightarrow [0, 1]$

let $Y_1, \dots, Y_k \in [0, 1]$

indep. and unifs. rand.

$$X = \min \{Y_1, \dots, Y_k\}$$

$$\text{then } E[X] = \frac{1}{k+1}$$

$$\text{and } E[X^2] = \frac{2}{(k+1)(k+2)}$$

$$\textcircled{1} X \leftarrow 1$$

② for each element e

$$\textcircled{a} X \leftarrow \min\{X, h(e)\}$$

$$/* E[X] = \frac{1}{k+1}, k = (\text{\# distinct elements}) */$$

$$\textcircled{3} \text{ return } \frac{1}{X} - 1$$

$$\text{Var}(X) = E[(X - E[X])^2]$$

$$X_1, \dots, X_k \text{ indep. copies of } X$$

$$\bar{X} = \frac{1}{k}(X_1 + \dots + X_k)$$

$$\text{Var}(\bar{X}) = \frac{1}{k} \text{Var}(X)$$

$$P[X_1 + \dots + X_k \leq (1 - \epsilon)k] \leq e^{-\epsilon^2 k/4}$$

Lecture 8

Probability

Gaussian random variables

Gaussian random variables

$$x \sim \mathcal{N}(\mu, \sigma^2)$$

mean variance

Def: the nicest, most pleasant, convenient random variable ever

Fact let $x \sim \mathcal{N}(\mu, \sigma^2)$, $d \in \mathbb{R}$.

$$\text{then } x+d \sim \mathcal{N}(\mu+d, \sigma^2)$$

$$dx \sim \mathcal{N}(d\mu, d^2\sigma^2)$$

Fact let $x_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $x_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$

$$\text{then } x_1 + x_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Fact let $x \sim \mathcal{N}(0, 1)$, $t < \frac{1}{2}$

$$\text{then } E[e^{tx^2}] = \frac{1}{\sqrt{1-2t}}$$

Algorithms

Dimensionality reduction

Bag-of-words

literally any sentence is fine

any ↑ is literally sentence English
five encodes text in high dimensions

Lemma 8.8. Let $A \sim \mathcal{N}^{k \times d}$ be a random matrix where each coordinate A_{ij} is an independently drawn sample from $\mathcal{N}$. Let $\epsilon > 0$ be sufficiently small. For any vector x ,

$$\mathbb{P}\left[(1-\epsilon)\|x\|^2 \leq \frac{1}{k}\|Ax\|^2 \leq (1+\epsilon)\|x\|^2\right] \geq 1 - 2e^{-k/8}$$

$$k \left\{ \left[\frac{1}{k} \|Ax\|^2 \right] \leftarrow \alpha_i \right.$$

$$E[\|Ax\|^2] = \sum_{i=1}^k E[(Ax)_i^2] = k\|x\|^2$$

$$E[(\sum_{i=1}^k A_{ij} x_j)^2] = \|x\|^2$$

$$A_{ij} x_j \sim \mathcal{N}(0, x_j^2)$$

$$\sum_{j=1}^n A_{ij} x_j \sim \mathcal{N}(0, \|x\|^2)$$

$$(Ax) \sim \mathcal{N}^k(0, \|x\|^2)$$

$$\Pr[\|Ax\|^2 \geq k\|x\|^2 \cdot (1+\epsilon)] \leq e^{-\epsilon^2 k/8}$$

Dimensionality Reduction - high level

$O(\log n)$ dimensions suffice

where $n = \#$ points

$$\mathbb{R}^{\text{very large}} \xrightarrow{f(x)} \mathbb{R}^{O(\log n)}$$

One can map n points into $\mathbb{R}^{O(\log n/\epsilon^2)}$ while preserving all distances up to $(1 \pm \epsilon)$ -APX

Lecture 9

Probability

Locality-sensitive hashing

Fix $\theta \in [0, \pi]$. given q , either:

(a) output $p \in P$ w/ $\angle(p, q) \leq 4\theta$

(b) declare no $p \in P$ w/ $\angle(p, q) \leq \theta$

define $h: S^{d-1} \rightarrow \{-1, 1\}^k$ by

$$h_i(x) = \text{sign}(\langle g_i, x \rangle) \text{ w/ } g_i \sim \mathcal{N}^d$$

when querying y :

• if there is $x \in P$ w/ $\angle(x, y) \leq \theta$, then it collides w/ probability $\geq 1/\sqrt{n}$

• expect to collide w/ at most one point $x \in P$ w/ $\angle(x, y) > 2\theta$

Lemma let $\bullet x, y \in S^{d-1}$
 $\bullet g \sim \mathcal{N}^d$
 then

$$P[\text{sign}(\langle g, x \rangle) \neq \text{sign}(\langle g, y \rangle)] = \frac{\angle(x, y)}{\pi}$$

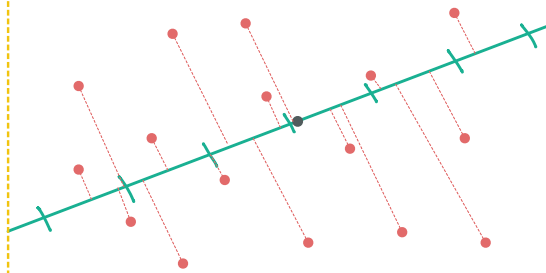
$$k = \pi(\log(n)/2\theta)$$



d-dimensional sphere

Random line embedding & Buckets

$$h(x) = \lfloor \langle g, x \rangle + d \rfloor \text{ w/ } g \sim \mathcal{N}^d, d \in [0, 1] \text{ unifs. rand.}$$

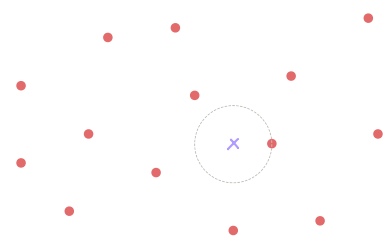


Algorithms

Approximate nearest neighbors

Nearest neighbor search

Input: n points $P \subset \mathbb{R}^d$



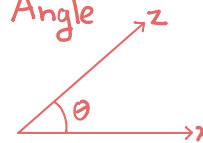
Query: given $q \in \mathbb{R}^d$, find closest point in P to q

Types of distances

• Euclidean distance

$$\|x - z\|_2 = \sqrt{\sum_i (x_i - z_i)^2}$$

• Angle



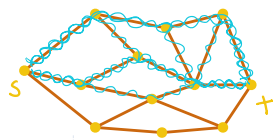
Lecture 10

Probability (and techniques)

LP duality

L_1 -metric embeddings

Max Flow Min Cut



Path Packing LP

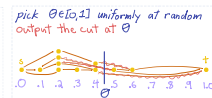
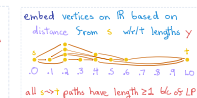
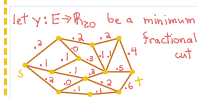
maximize $\sum_{p \in P} x_p$
over $x: P_{st} \rightarrow \mathbb{R}_{\geq 0}$
(all s-t paths)

s.t. $\sum_{p \in P} x_p \leq c_e$ for all edges e
"Ax ≤ c" (capacity of e)

Path Covering LP (i.e. cut)

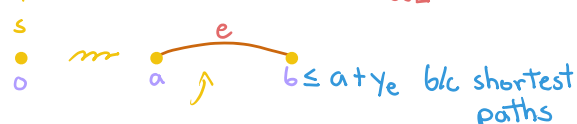
minimize $\sum_{e \in E} c_e y_e$
over $y: E \rightarrow \mathbb{R}_{\geq 0}$
(total capacity of "fractional cut")

s.t. $\sum_{e \in P} y_e \geq 1$ for all $p \in P_{st}$
"A'y ≥ b" (continuous analog of s-t cut)



$E[\text{capacity of cut}]$

$$\stackrel{\text{(linearity of expectation)}}{=} \sum_{e \in E} c_e \Pr[e \text{ cut}] \leq \sum_{e \in E} c_e y_e$$

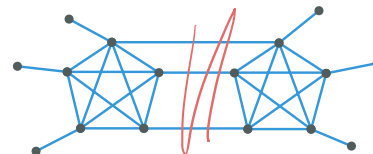


Algorithms

Sparsest Cut

Special case: uniform sparsest cut
w/ uniform demands $b(u,v) = 1 \quad \forall u,v$

$$\text{"sparsity"} = \frac{c(\delta(s))}{|S| |T|}$$



useful primitive for graph partitioning, divide & conquer

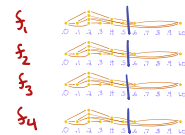
Q: how to round LP to sparse cut?

min $\sum_{e \in E} c_e d(u,v)$ over metrics $d: V \times V \rightarrow \mathbb{R}_{\geq 0}$
s.t. $\sum_{i=1}^k b_i d(s_i, t_i) \geq 1$

Suppose $d(u,v)$ was of the form

$$d(u,v) = \sum_{i=1}^k |\xi_i(u) - \xi_i(v)| \quad \text{for } \xi_i: V \rightarrow [0,1]^k$$

(assume $\min \xi_i(a) = 0, \max \xi_i(a) = 1$)



" L_1 -metric"

Lecture 11

Probability (and techniques)

Algorithms

Other metric-based problems
(sometimes via correspondence between cuts $\leftrightarrow$ metrics)

Buy-at-bulk network design
k-median clustering
metric labeling
group Steiner tree
metrical task systems
vehicle routing
hierarchical placement
distributed k-server
covering Steiner tree
⋮



these problems are easy for tree metrics

Approximation by tree metrics

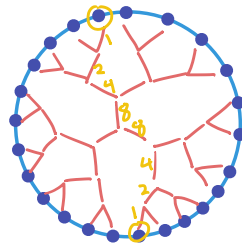
Input: undirected $G=(V,E)$ w/ edge weights $w: E \rightarrow \mathbb{R}_{\geq 0}$

(High-level)
Goal: compute tree T (containing V as nodes)
so that shortest paths on T are close to T

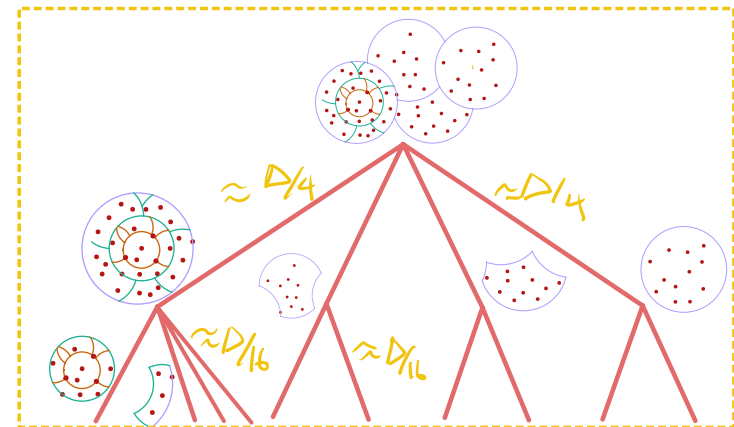
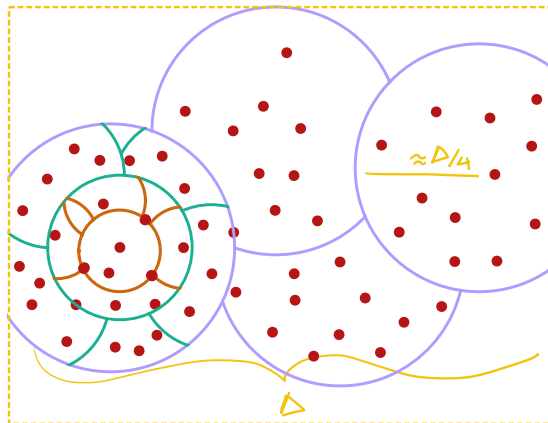
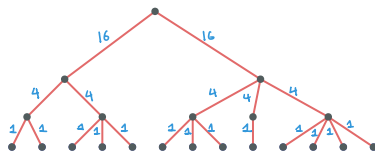
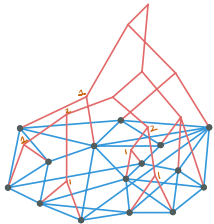
$$d_T(u,v) \geq d_G(u,v)$$

$$\text{For } e=f_u,v \quad \begin{matrix} \text{(distance in } T) \\ \text{(stretch } e) = \frac{d_T(u,v)}{d_G(u,v)} \\ \text{(distance in } G) \end{matrix}$$

ideally
(stretch e) = $\text{polylog}(n) \forall e$



Theorem In randomized polynomial time,
randomized hierarchical dominating tree metric w/
 $E[\text{stretch } e] \leq O(\log n) \quad \forall e \in E$



Lecture 12

Probability (and techniques)

Additive Chernoff Inequality

Double-sampling

VC dimension

Additive Chernoff bound

let $X_1, \dots, X_n \in [0, 1]$ be independent

$$\mu = E[X_1 + \dots + X_n] / n$$

$\mu \in (0, 1)$

- $P\left[\frac{X_1 + \dots + X_n}{n} \geq \mu + \epsilon\right] \leq e^{-2\epsilon^2 n}$
- $P\left[\frac{X_1 + \dots + X_n}{n} \leq \mu - \epsilon\right] \leq e^{-2\epsilon^2 n}$

Algorithms

ϵ -samples and ϵ -nets

ϵ -sample $|M_Q(r) - \mu(r)| \leq \epsilon \quad \forall r \in R$

let Q sample k points w/ repetition
 $k = O\left(\frac{\ln(g(|P|))}{\epsilon^2}\right) \Rightarrow \epsilon$ -sample $k = O\left(\frac{\ln(g(|P|))}{\epsilon^2}\right) \Rightarrow \epsilon$ -net (w/ constant probability)

ϵ -sample theorem [VC71]
 $k \geq O\left(\frac{\ln(g(2k))}{\epsilon^2}\right) \Rightarrow \epsilon$ -sample (w/ constant probability)

e.g. $g(n) = O(n^d)$
 $k \geq O\left(\frac{d \ln(d/\epsilon)}{\epsilon^2}\right) \Rightarrow \epsilon$ -sample
 ↑ independent of $|P|$ (!)

$P\left[\frac{X_1 + \dots + X_n}{n} \geq \mu + \epsilon\right], P\left[\frac{X_1 + \dots + X_n}{n} \leq \mu - \epsilon\right] \leq e^{-2\epsilon^2 n}$ $g(n) = \max\{|RAQ| : a \in P, |a| \leq n\}$

$\mu(r) = \frac{|P \cap r|}{|P|}$
 $M_Q(r) = \frac{|a \cap r|}{|a|}$
 $RAQ = \{r \cap a : r \in R\}$

ϵ -sample $|M_Q(r) - \mu(r)| \leq \epsilon \quad \forall r \in R$

let Q sample k points w/ repetition
 $k = O\left(\frac{\ln(g(|P|))}{\epsilon^2}\right) \Rightarrow \epsilon$ -sample $k = O\left(\frac{\ln(g(|P|))}{\epsilon^2}\right) \Rightarrow \epsilon$ -net (w/ constant probability)

ϵ -net theorem [HW87]
 $k \geq O\left(\frac{\ln(g(2k))}{\epsilon^2}\right) \Rightarrow \epsilon$ -net (w/ constant probability)

e.g. $g(n) = O(n^d)$
 $k \geq O\left(\frac{d \ln(d/\epsilon)}{\epsilon^2}\right) \Rightarrow \epsilon$ -net
 ↑ independent of $|P|$ (!)

$P\left[\frac{X_1 + \dots + X_n}{n} \geq \mu + \epsilon\right], P\left[\frac{X_1 + \dots + X_n}{n} \leq \mu - \epsilon\right] \leq e^{-2\epsilon^2 n}$ $g(n) = \max\{|RAQ| : a \in P, |a| \leq n\}$

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 $RAQ = \{r \cap a : r \in R\}$

Lecture 13

Probability
(and techniques)

Algorithms
(models and applications)

Consistency model

PAC learning model

Generalization error

Occam's Razor

Generalization by growth

Lecture 14

Probability
(and techniques)

Algorithms
(models and applications)

Entropy

(Optimal)
Error-Correcting
Codes

Lecture 15

Probability (and techniques)

Lovász Local Lemma

Algorithms (models and applications)

Moser's resampling algorithm

Lecture 16

Probability (and techniques)

Algorithms (models and applications)

Random Walks

Stationary Distribution

Perron-Frobenius Theorem

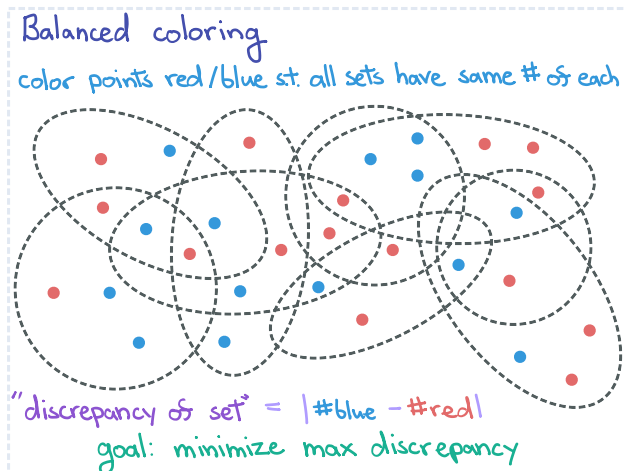
Pagerank

Lecture 17

Probability (and techniques)

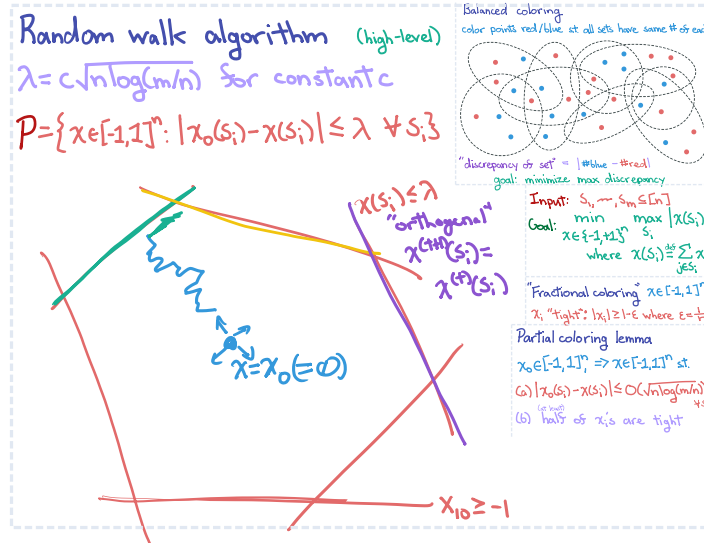
High-dimensional random walks

More fun w/ Gaussians



Algorithms (models and applications)

Discrepancy minimization



Lecture 19

Probability (and techniques)

Algorithms (models and applications)

Spectral theorem
for symmetric maps

Spectral analysis of
random walks

Lecture 20

Probability
(and techniques)

Algorithms
(models and applications)

Conductance

Cheeger's inequality

Conductance-based
mixing time

Fiedler's algorithm
for low-conductance
cuts

Lecture 21

Probability
(and techniques)

Algorithms
(models and applications)

Expanders

Zigzag product

tensors

Deterministic

Connectivity in

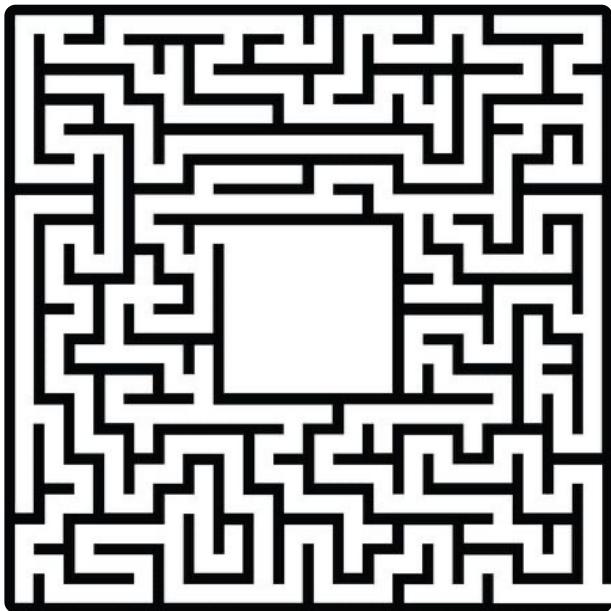
$O(\log n)$ space

connectivity

Input: $G=(V,E)$ undirected,
(m edges, n vertices)

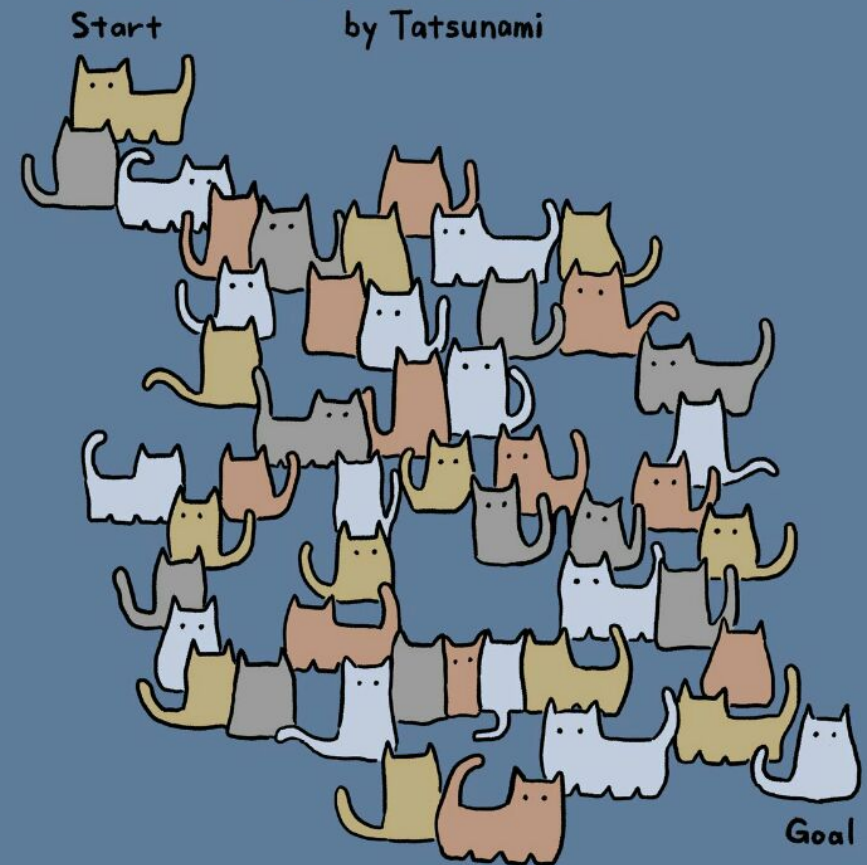
$s, t \in V$

Q: is there a path from
s to t?



CAT MAZE

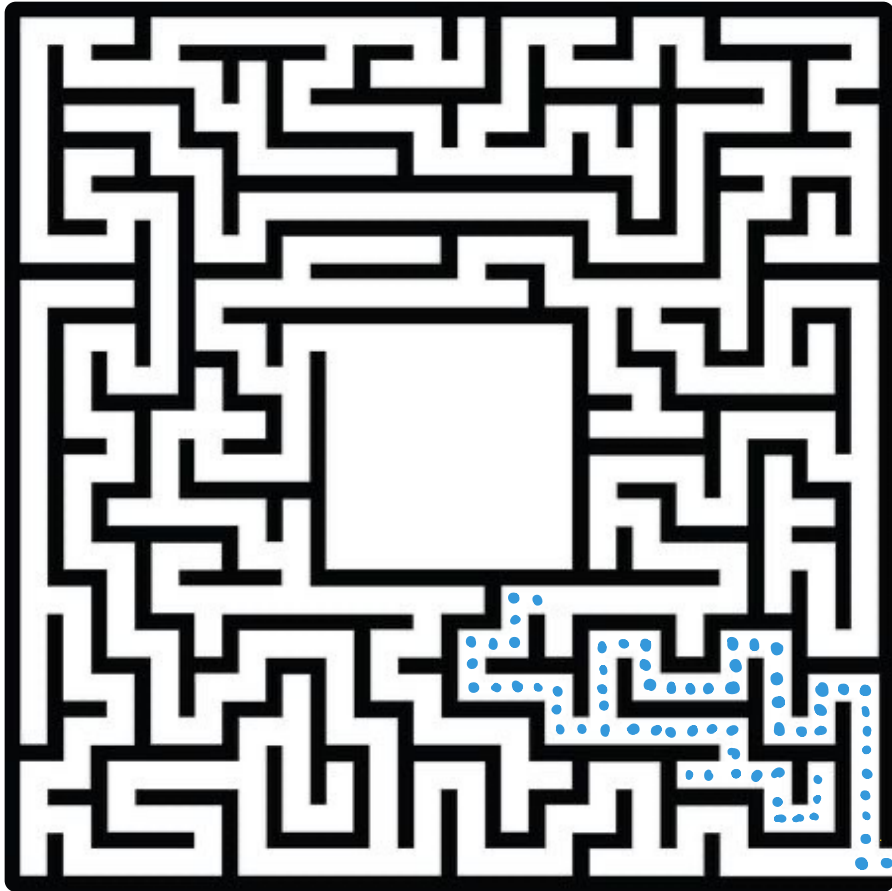
by Tatsunami



Reach the Goal by following the cats, alternating between cats facing the front and cats facing backwards.

The cats in your path must be connected or overlapping.

It doesn't matter what color or size they are.



Breadcrumbs

BFS, DFS take
linear time and space

can connectivity be solved in...

Lies

Yes

No

Not
voting

linear space?

BFS,
DFS

$O(\log^{O(1)} n)$ space?

90%

1

3

$O(\log n)$ space?

90%

4

2

$O(\log n)$ space
deterministically?

50%

50%

2 weeks
from now

can connectivity be solved in $O(\log n)$ space?

algorithm:

- (1) repeat $O(\log n)$ times:
 - (a) start at $s \in V$
 - (b) randomly walk for $O(mn)$ steps.
 - (c) if walk visits t , return TRUE
- (2) return False

random walk terminology:

Hitting time $H(s, t)$ = expected # steps until a random walk from s reaches t

Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex

Theorem

If G is connected, then $C(s) \leq m(n-1)$

Q: how about w/out randomization?

Recall

$$\text{score}_3(v) = \sum_{(u,v) \in E} \frac{\text{score}_3(u)}{\deg^+(u)}$$



let $\Delta^V = \{x \in \mathbb{R}_V^{\geq 0} : \sum_v x_v = 1\} = \{\text{distributions over } V\}$

Theorem there exists $x \in \Delta^V$ s.t.

$$x_v = \sum_{(u,v) \in E} \frac{x_u}{\deg^+(u)} \quad \forall v \in V.$$

If G is strongly connected, then x is unique and strictly positive.

Theorem, restated: every random walk has a stationary distribution. If G is strongly connected, then this distribution is unique.

Theorem, restated: let G be a dir. graph and R the random walk linear map. Then R has an eigenvector $x \in \Delta^V$ w/ eigenvalue 1. If G is strongly connected, then x is the unique eigenvector w/ eigenvalue 1

Recall

Theorem: $G=(V,E)$ undirected, $w:E \rightarrow \mathbb{R}_{>0}$
 let $R:\mathbb{R}^V \rightarrow \mathbb{R}^V$ be random walk map
 then R has stationary distr. $s \in \Delta_V$ w/

$$s_v = \frac{\deg(v)}{2W}$$

$$\deg(v) = \sum_{e \in \mathcal{E}(v)} w(e)$$

$$W = \sum_{e \in E} w(e)$$

Proof $d \in \mathbb{R}^V$ be the degrees

$$D = \text{diag}(d)$$

$$A = \text{adj. matrix} \in \mathbb{R}^{n \times n}$$

$$R = AD^{-1}$$

$$Rd = AD^{-1}d = A\mathbf{1} = d$$

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) \rangle = \langle x, u \rangle \langle v, x \rangle$

Laplacian of $e = \{u, v\}$
 $L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{V \times V}$
 $(1_u = \{0,1\}$ -indicator vector for u)
 $\langle x, L_e x \rangle = (x_u - x_v)^2$

Laplacian of 
 $L = \sum_{e \in E} w(e) L_e$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

positive semi-definite A :
 • A is symmetric
 • $\langle x, Ax \rangle \geq 0 \quad \forall x$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ maximizes $\langle x, Lx \rangle$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$
 where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

Recall

$$Q = \frac{\sqrt{d} \otimes \sqrt{d}}{\langle \mathbf{1}, d \rangle} + \mu_2 (v_2 \otimes v_2) + \dots + \mu_n (v_n \otimes v_n)$$

w/ $\mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n$
(also eigenvalues of R)

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary distribution

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) \rangle = \langle x, u \rangle \langle v, x \rangle$

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where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

Theorem: $G=(V, E)$ undirected, $w: E \rightarrow \mathbb{R}_{>0}$
then R has stationary distr. $s \in \Delta_V$ w/

$$s_v = \frac{\deg(v)}{2W} \quad \deg(v) = \sum_{e \in \partial(v)} w(e), \quad W = \sum_{e \in E} w(e)$$

Lemma let $A = C^{-1}BC$.

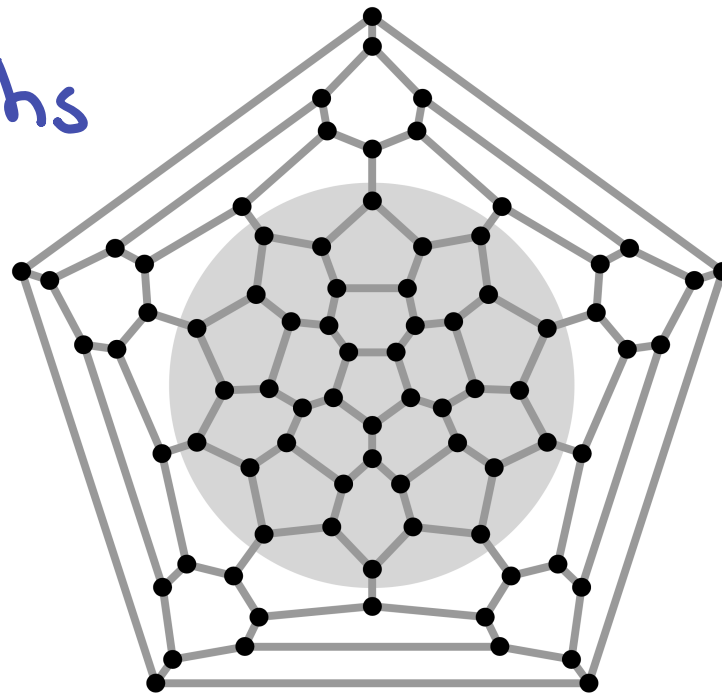
- A and B have same eigenvalues
- C maps eigenvectors of A to eigenvectors of B

Normalized random walk matrix

$$Q = D^{-1/2} R D^{1/2} = D^{-1/2} A D^{-1/2}$$

- $D = \text{diag}(d)$ w/ $d = \text{degrees}$
- $A = \text{adjacency matrix}$
- $R = A D^{-1} = \text{random walk matrix}$
- Q is symmetric!
- Q has same eigenvalues as R !

Expander graphs



Theorem, restated: let G be a dir. graph and R the random walk linear map. Then R has an eigenvector $x \in \Delta^V$ w/ eigenvalue 1. If G is strongly connected, then x is the unique eigenvector w/ eigenvalue 1

Theorem: $G=(V,E)$ undirected, $w:E \rightarrow \mathbb{R}_{>0}$
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Theorem. let $\delta = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1-\delta)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$
stationary distribution
 $1 \geq \mu_2 \geq \dots \geq \mu_n = \text{eigenvalues of } R$

G is an expander if the spectral gap δ is at least a constant

Q: deterministic connectivity for constant-degree expanders?

Q: diameter of a constant-degree expander?

Q: diameter of a
constant-degree expander?

say $\gamma = 1/2$. let $x = 1_s$ for $s \in V$

$$\|R^k x - \underset{\substack{\uparrow \\ \text{stationary} \\ \text{distribution}}}{\frac{d}{2m}}}\|_2 \leq \left(\frac{1}{2}\right)^k \cdot n$$

for $k = O(\log n)$,

$$\|R^k x - \frac{d}{2m}\|_\infty \leq \|R^k x - \frac{d}{2m}\|_2 \leq \frac{1}{\text{poly}(n)}$$

$$\Rightarrow (R^k x)_v \geq \frac{d}{2m} - \frac{1}{\text{poly}(n)} > 0 \quad \forall v$$

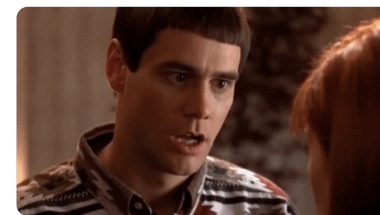
$\Rightarrow$ there exists a path of $O(\log n)$ edges from s to t

Theorem, restated: let G be a dir. graph and R the random walk linear map. Then R has an eigenvector $x \in \Delta^V$ w/ eigenvalue 1. If G is strongly connected, then x is the unique eigenvector w/ eigenvalue 1

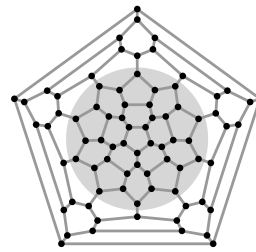
Theorem: $G=(V,E)$ undirected, $w:E \rightarrow \mathbb{R}_{>0}$
let $R:\mathbb{R}^V \rightarrow \mathbb{R}^V$ be random walk map
then R has stationary distr. $s \in \Delta_V$ w/
 $s_v = \frac{\deg(v)}{2W}$ $\deg(v) = \sum_{e \in \partial(v)} w(e)$
 $W = \sum_{e \in E} w(e)$

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$
 $\|R^k x - s\| \leq (1-\gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$
 $\uparrow$
stationary distribution
 $1 \geq \mu_2 \geq \dots \geq \mu_n = \text{eigenvalues of } R$

G is an expander if the spectral gap γ is at least a constant



Expander graphs



G is an expander if the spectral gap γ is at least a constant

Theorem, restated: let G be a dir. graph and R the random walk linear map. Then R has an eigenvector $x \in \Delta^V$ w/ eigenvalue 1. If G is strongly connected, then x is the unique eigenvector w/ eigenvalue 1

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 $1 \geq \mu_2 \geq \dots \geq \mu_n = \text{eigenvalues of } R$

Q: deterministic connectivity for constant-degree expanders?

✓ Q: diameter of a constant-degree expander?
 $O(\log n)$

Q: how many random bits are needed for 1 step of a random walk in a constant-degree graph?

Expander graphs

G is an expander if the spectral gap γ is at least a constant

Theorem, restated: let G be a dir. graph and R the random walk linear map. Then R has an eigenvector $x \in \Delta^V$ w/ eigenvalue 1. If G is strongly connected, then x is the unique eigenvector w/ eigenvalue 1

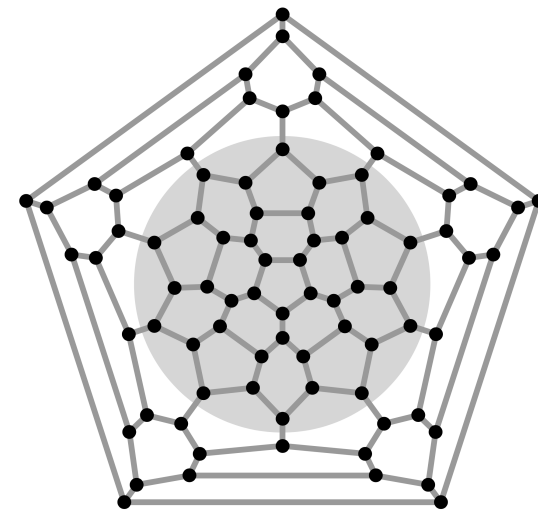
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Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$
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stationary distribution
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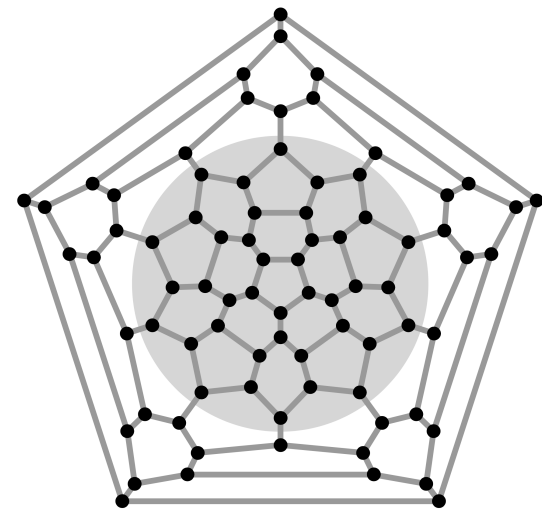
Q: deterministic connectivity for constant-degree expanders?

- Diameter $O(\log n)$
- $O(1)$ random bits per random step

$\Rightarrow$ enumerate all paths of length $O(\log n)$!



Lemma: $O(\log n)$ space, poly-time
deterministic connectivity in
 $O(1)$ -degree expanders



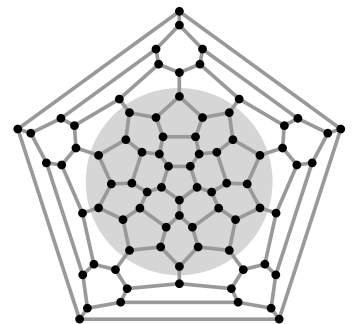
[Reingold, 2005]

Theorem: $O(\log n)$ space, poly-time
deterministic connectivity in undirected graphs

Theorem: $O(\log n)$ space, poly-time [Reingold, 2005]
deterministic connectivity in undirected graphs.

High level idea:

- convert G into $O(1)$ -degree expander
by sequence of "graph transformations"
(graph transformations applied implicitly due to space constraints)
- derandomize the random walk
algo (implicitly) on the expander

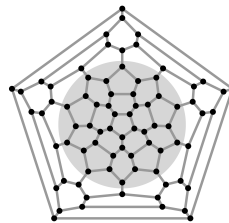


Lemma: $O(\log n)$ space, poly-time
deterministic connectivity in
 $O(1)$ -degree expanders

Theorem: $O(\log n)$ space, poly-time [Reingold, 2005]
deterministic connectivity in undirected graphs.

High level idea:

- Convert G into $O(1)$ -degree expander by sequence of "graph transformations"
(graph transformations applied implicitly due to space constraints)
- derandomize the random walk algo (implicitly) on the expander



Lemma: $O(\log n)$ space, poly-time
deterministic connectivity in
 $O(1)$ -degree expanders

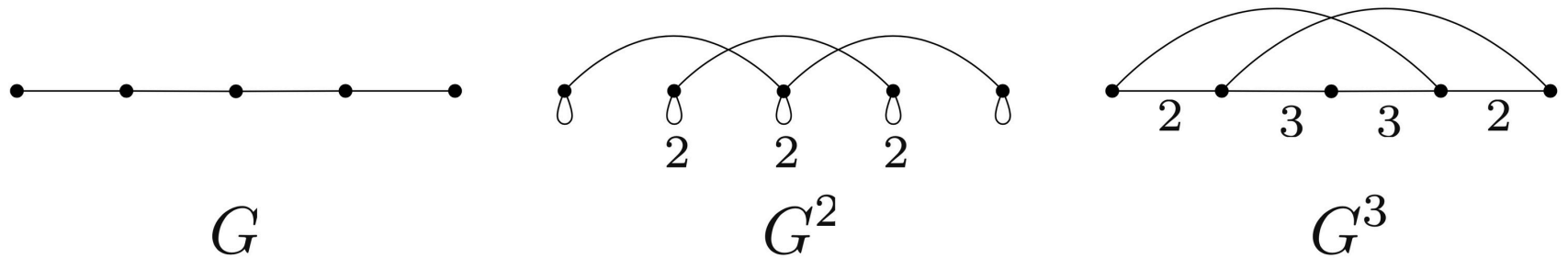
Graph transformations:

1. Powering
2. Zig-Zag product

Preliminary

WLOG assume G is a regular graph
of degree $d = O(1)$

Powering



$G^k =$ graph w/ an edge for every k step walk in G
(w/ repetition)

random walk P^k has eigenvalues $1, \mu_2^k, \dots, \mu_n^k$

$$\delta(G^k) = 1 - (1 - \delta(G))^k \quad \text{😊}$$

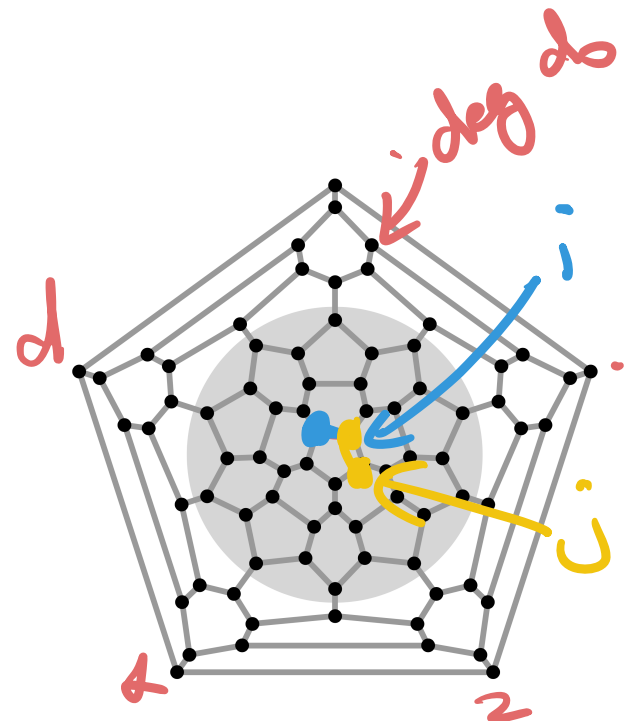
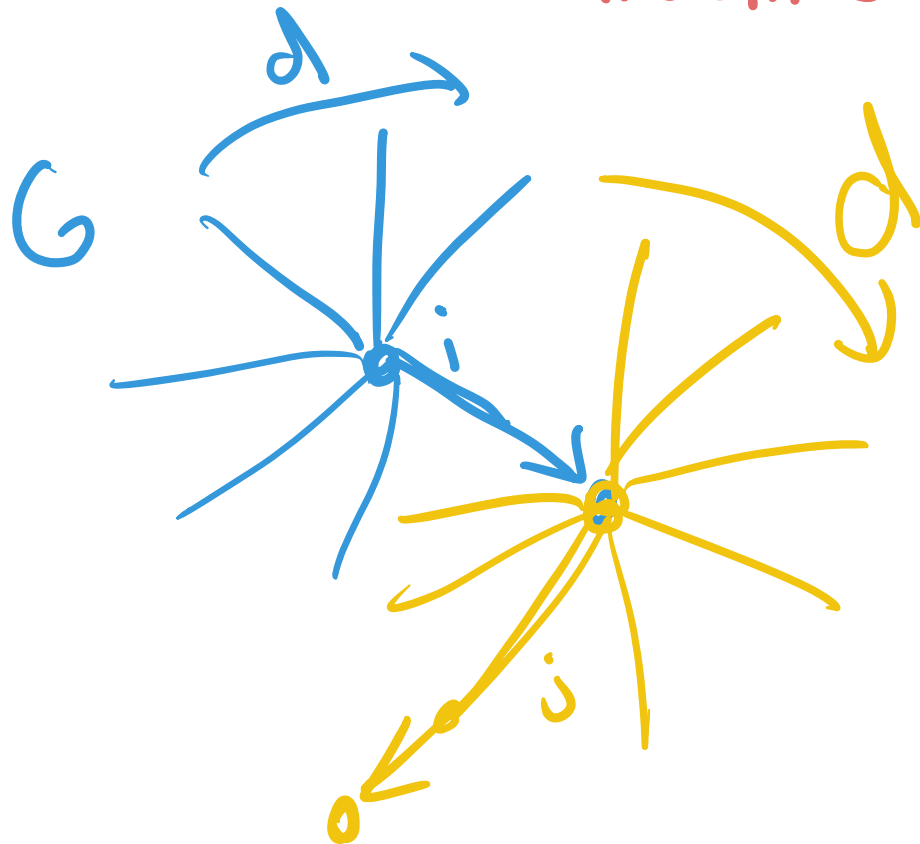
degree increases from d to d^k 😞

Zig-zag product $Z(G, H)$

let G be d -regular

Problem: d large (after powering)

how to simulate "random step" w/ fewer bits?



Zig-zag product $Z(G, H)$

let G be d -regular

H be d_0 -regular w/ d vertices (H will be $O(D)$ -degree expander)

vertices: $V_G \times V_H = \{(u, v) : u \in V_G, v \in V_H\}$

edges: for $(v_1, i_1) \in V_G \times V_H$, $k_1, k_2 \in [d_0]$:

the (k_1, k_2) th neighbor of (v_1, i_1) is (v_2, i_4) where:

$$\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} \xrightarrow{(a)} \begin{pmatrix} v_1 \\ i_2 \end{pmatrix} \xrightarrow{(b)} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix} \xrightarrow{(c)} \begin{pmatrix} v_2 \\ i_3 \end{pmatrix} \xrightarrow{(d)} \begin{pmatrix} v_2 \\ i_4 \end{pmatrix}$$

where:

- (a) is a step in H from i_1 to its k_1 th neighbor, i_2 .
- (b) moves v_1 to the i_2 th outgoing neighbor of v_1 , v_2 .
- (c) moves from i_2 to i_3 where v_1 is the i_3 th neighbor of v_2 .
- (d) moves i_3 to its k_2 th neighbor in H .

(so $Z(G, H)$ is d_0^2 -regular)

Lemma: $Z(G|H)$ has

- n vertices
- degree d_0^2 😊
- spectral gap $\gamma_G \gamma_H^2$ 😊

Zig-zag product $Z(G,H)$

let G be d -regular

H be d_0 -regular w/ d vertices (H will be $O(1)$ -degree expander)

vertices: $V_G \times V_H = \{(u,v) : u \in V_G, v \in V_H\}$

edges: for $(v_1, i_1) \in V_G \times V_H$, $k_1, k_2 \in [d_0]$:

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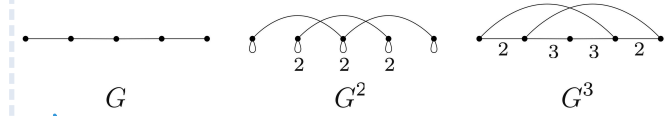
- (a) is a step in H from i_1 to its k_1 th neighbor, i_2 .
- (b) moves v_1 to the i_2 th outgoing neighbor of v_1 , v_2 .
- (c) moves from i_2 to i_3 where v_1 is the i_3 th neighbor of v_2 .
- (d) moves i_3 to its k_2 th neighbor in H .

(so $Z(G,H)$ is d_0^2 -regular)

	G^k	$Z(G, H)$
vertices	n	nd
degree	d^k	d_0^2
spectral gap	$1 - (1 - \gamma_G)^k$	$\gamma_G \gamma_H^2$

	$Z(G^2, H)$
vertices	nd^2
degree	d_0^2
spectral gap	$(1 - (1 - \gamma_G^2)) \gamma_H^2 = (2\gamma_G - \gamma_G^2) \cdot \gamma_H^2$ $= 2\gamma_G \gamma_H^2 - \gamma_G^2 \gamma_H^2$

Powering



G^k = graph w/ edge for every k step walk in G
(w/ repetition)

random walk R^k has eigenvalues $1, \mu_1^k, \dots, \mu_n^k$

$$\gamma(G^k) = 1 - (1 - \gamma(G))^k \quad \text{😊}$$

degree increases from d to d^k 😞

Zig-zag product $Z(G, H)$

let G be d -regular

H be d_0 -regular w/ d vertices (H will be $O(d)$ -degree expander)

vertices: $V_G \times V_H = \{(u, v) : u \in V_G, v \in V_H\}$

edges: for $(v_1, i_1) \in V_G \times V_H$, $k_1, k_2 \in [d_0]$:

the (k_1, k_2) th neighbor of (v_1, i_1) is (v_2, i_2) where:

$$\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} \xrightarrow{(a)} \begin{pmatrix} v_1 \\ i_2 \end{pmatrix} \xrightarrow{(b)} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix} \xrightarrow{(c)} \begin{pmatrix} v_2 \\ i_3 \end{pmatrix} \xrightarrow{(d)} \begin{pmatrix} v_2 \\ i_4 \end{pmatrix}$$

where:

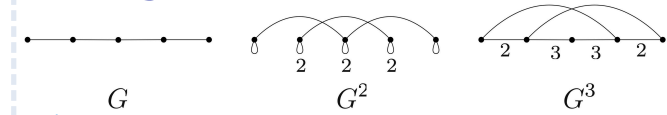
- (a) is a step in H from i_1 to its k_1 th neighbor, i_2 .
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(so $Z(G, H)$ is d_0^2 -regular)

	G^k	$Z(G, H)$
vertices	n	nd
degree	d^k	d_0^2
spectral gap	$1 - (1 - \gamma_G)^k$	$\gamma_G \gamma_H^2$

	$Z(G^2, H)$
vertices	nd
degree	d_0^2
spectral gap	$(1 - (1 - \gamma_G^2)) \gamma_H^2$

Powering



G^k = graph w/ edge for every k step walk in G
(w/ repetition)

random walk R^k has eigenvalues $1, u_1^k, \dots, u_n^k$

$$\gamma(G^k) = 1 - (1 - \gamma(G))^k \quad \text{😊}$$

degree increases from d to d^k 😞

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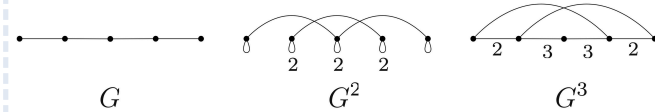
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	$Z(G^2, H)$
vertices	nd
degree	d_0^2
spectral gap	$(1 - (1 - \gamma_G)^2) \gamma_H^2$

for $\gamma_G \leq \frac{1}{16}$, $\gamma_H \geq 3/4$,

$$(1 - (1 - \gamma_G)^2) \gamma_H^2 \geq 1.01 \gamma_G$$

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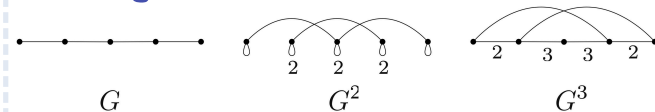
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degree	d^k	d_0^2
spectral gap	$1 - (1 - \gamma_G)^k$	$\gamma_G \gamma_H^2$

Space usage

for graphs like G^k or $Z(G, H)$,
need to answer in $O(1)$ space:

Powering



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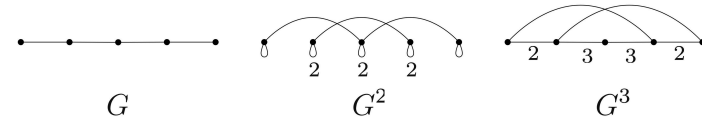
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		$Z(G^2, H)$
vertices		nd
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(i_1, i_2) th neighbor of v in G^2 ?

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(k_1, k_2) th neighbor of (v_1, i_1) ?

Zig-zag product $Z(G, H)$

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Analysis of zig-zag product

let R_H = random walk for H

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(so $Z(G, H)$ is d^2 -regular)

write random walk for $Z(G, H)$ as

$$(I \otimes R_H) Z (I \otimes R_H)$$

where:

- $(I \otimes R_H)$ takes random step in H ,
leaves G -coordinate fixed
- Z does deterministic "zig-zag" step

Simpler zig-zag product

$$\text{let } S = (I \otimes 1)/d$$

= uniformly sample 1 vertex in H

consider

$$(I \otimes S)Z(I \otimes S)$$

where:

- $(I \otimes S)$ randomly samples vertex in H
- Z uses H -coordinate to take step in G

Zig-zag product $Z(G, H)$

let G be d -regular

(H will be $O(d)$ -degree expander)

H be d_0 -regular w/ d vertices

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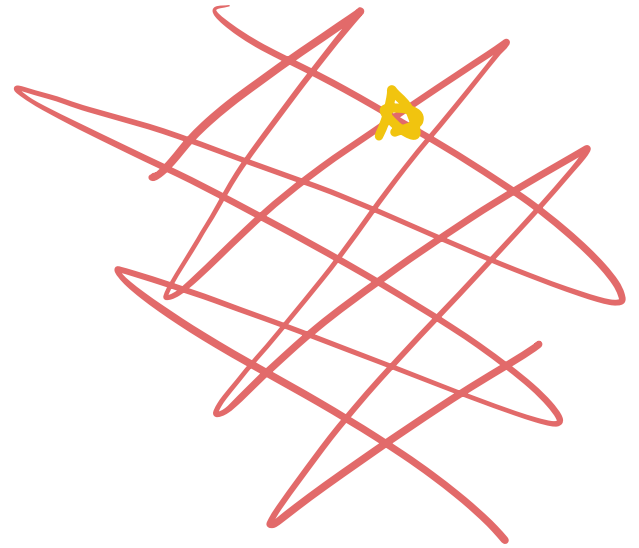
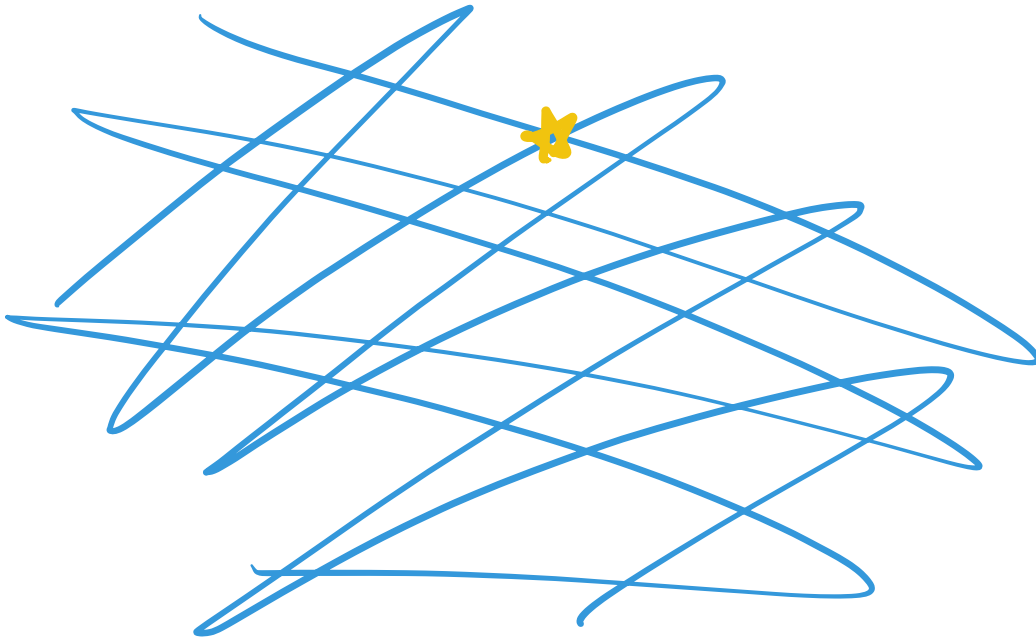
Idea: if H expander, then random walk on H behaves similarly to random sample.

$$\Rightarrow (I \otimes R_H) Z (I \otimes R_H) \approx \underbrace{(I \otimes S) Z (I \otimes S)}_{\text{good spectral gap}}$$

Tensor products of graphs

$$G_1 \otimes G_2 = (V_1 \times V_2, E_1 \times E_2)$$

$$E_1 \times E_2 = \{((u_1, u_2), (v_1, v_2)) : (u_1, v_1) \in E_1, (u_2, v_2) \in E_2\}$$



Tensor of vectors: Rank-1 matrix

$$u \otimes v \quad \text{where } u, v \in \mathbb{R}^n$$

(aka. uv^T)

defined by $(u \otimes v)y = \langle v, y \rangle u$

as a matrix:

$$u \otimes v = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_n v_1 & u_n v_2 & & u_n v_n \end{bmatrix}$$

$$\otimes \text{ maps } u, v \in \mathbb{R}^n \rightarrow u \otimes v \in \mathbb{R}^{n \times n} = \mathbb{R}^n \rightarrow \mathbb{R}^n$$

Tensor products

for $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{n \times n}$,

$A \otimes B$ is a linear map on $\mathbb{R}^{m \times n}$ defined by

$$(A \otimes B)C = ACB^T \quad \text{for } C \in \mathbb{R}^{m \times n}$$

Tensor product $I \otimes R_H$

$I \otimes R_H$ is a linear map on $\mathbb{R}^{V \times d}$ defined by

$$((I \otimes R_H)X)_v = R_H X_v \quad X \in \mathbb{R}^{V \times d}$$

where $X_v = v$ th row of X .

$$R_H = v_1 \otimes v_1 + \mu_2(v_2 \otimes v_2) + \dots + \mu_n(v_n \otimes v_n)$$

Zig-zag product $Z(G, H)$

let G be d -regular

H be d_0 -regular w/ d vertices (H will be $O(d)$ -degree expander)

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$$I \otimes (A + B) = I \otimes A + I \otimes B$$

$R'_H = R_H - \gamma_H S$ has eigenvalues in $[\gamma_H - 1, 1 - \gamma_H]$

$$(I \otimes R_H) Z (I \otimes R_H)$$

$$= (I \otimes (R'_H + \gamma_H S)) Z (I \otimes (R'_H + \gamma_H S))$$

=

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$$+ \gamma_H (I \otimes S) Z (I \otimes R'_H)$$

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Operator norm:

$$\|A\| = \max \{ \|Ax\| : \|x\| = 1 \}$$

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$$\|A + B\| \leq \|A\| + \|B\|$$

$$\|AB\| \leq \|A\| \|B\|$$

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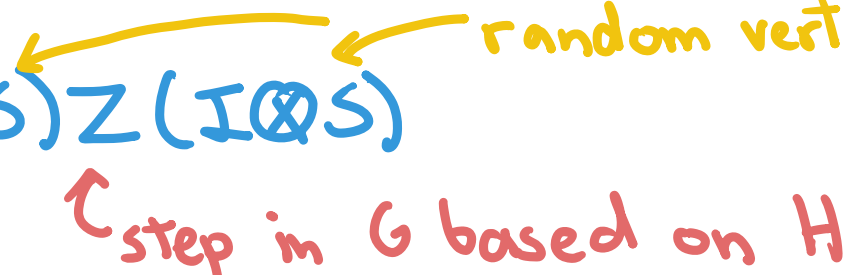
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$$+ \gamma_H (I \otimes R'_H) Z (I \otimes S)$$

$$+ (I \otimes R'_H) Z (I \otimes R'_H)$$

$(I \otimes S) Z (I \otimes S)$


$\Rightarrow$

- random step in G
- random vertex in H

 $\left. \vphantom{\begin{matrix} \bullet \text{ random step in } G \\ \bullet \text{ random vertex in } H \end{matrix}} \right\} R_G \otimes S$

Zig-zag product $Z(G, H)$

let G be d -regular

(H will be $O(d)$ -degree expander)

H be d_0 -regular w/ d vertices

vertices: $V_G \times V_H = \{(u, v) : u \in V_G, v \in V_H\}$

edges: for $(v_1, i_1) \in V_G \times V_H$, $k_1, k_2 \in [d_0]$:

the (k_1, k_2) th neighbor of (v_1, i_1) is (v_2, i_2)

$$\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} \xrightarrow{(a)} \begin{pmatrix} v_1 \\ i_2 \end{pmatrix} \xrightarrow{(b)} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix} \xrightarrow{(c)} \begin{pmatrix} v_2 \\ i_3 \end{pmatrix} \xrightarrow{(d)} \begin{pmatrix} v_2 \\ i_4 \end{pmatrix}$$

where:

- (a) is a step in H from i_1 to its k_1 th neighbor, i_2 .
- (b) moves v_1 to the i_2 th outgoing neighbor of v_1 , v_2 .
- (c) moves from i_2 to i_3 where v_1 is the i_3 th neighbor of v_2 .
- (d) moves i_3 to its k_2 th neighbor in H .

(so $Z(G, H)$ is d^2 -regular)

let R_H = random walk for H

write random walk for $Z(G, H)$

as $(I \otimes R_H) Z (I \otimes R_H)$ where:

- $(I \otimes R_H)$ takes random step in H , leaves G -coordinate fixed
- Z does deterministic "zig-zag" step

Tensor product $I \otimes R_H$

$I \otimes R_H$ is linear map on $\mathbb{R}^{V_G \times d}$

$$((I \otimes R_H)X)_v = R_H X_v$$

where $X_v = v$ th row of X .

$$I \otimes (A+B) = I \otimes A + I \otimes B$$

$$\|A+B\| \leq \|A\| + \|B\|$$

$$\|AB\| \leq \|A\| \|B\|$$

$$\|A \otimes B\| \leq \|A\| \|B\|$$

$R'_H = R_H - \gamma_H S$ has eigenvalues in $[\gamma_H - 1, 1 - \gamma_H]$

$$(I \otimes R_H) Z (I \otimes R_H)$$

$$= \gamma_H^2 (I \otimes S) Z (I \otimes S)$$

$$+ \gamma_H (I \otimes S) Z (I \otimes R'_H)$$

$$+ \gamma_H (I \otimes R'_H) Z (I \otimes S)$$

$$+ (I \otimes R'_H) Z (I \otimes R'_H)$$

$$(I \otimes S) Z (I \otimes S) = R \otimes S$$

Fact: $A \otimes B$ has eigenvalues $\lambda_A \lambda_B$ where

λ_A is an eigenvalue of A ,

λ_B is an eigenvalue of B

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$$\|AB\| \leq \|A\| \|B\|$$

$$\|A \otimes B\| \leq \|A\| \|B\|$$

All put together:

For $x \in \mathbb{R}^{v \times d}$ orthogonal to $\mathbb{1}^{v \times d}$, $\|x\|=1$,

$$\langle x, (I \otimes R_H) Z (I \otimes R_H) x \rangle$$

$$= \gamma_H^2 \langle x, (I \otimes S) Z (I \otimes S) x \rangle$$

$$+ \gamma_H \langle x, (I \otimes S) Z (I \otimes R'_H) x \rangle$$

$$+ \gamma_H \langle x, (I \otimes R'_H) Z (I \otimes S) x \rangle$$

$$+ \langle x, (I \otimes R'_H) Z (I \otimes R'_H) x \rangle$$