

Deterministic connectivity
in logarithmic space

Lecture 16

Probability (and techniques)

Algorithms (models and applications)

Random Walks

Stationary Distribution

Perron-Frobenius Theorem

Pagerank

Lecture 18

Probability (and techniques)

Algorithms (models and applications)

Hitting times

Cover times

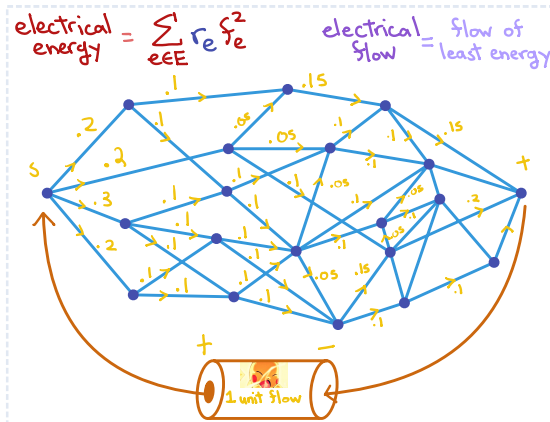
Electrical networks

Connectivity
in $O(\log n)$ space

random walk terminology:

Hitting time $H(s, t)$ = expected # steps until a random walk from s reaches t

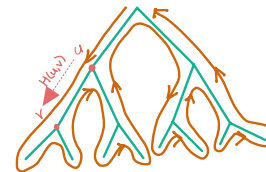
Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex



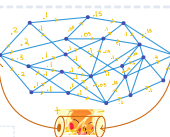
Lemma $H(u, v) + H(v, u) = 2m \cdot (\text{energy of unit flow } (u, v))$

Theorem $C(s) \leq 2mn$

proof Six traversal of spanning tree



$$\begin{aligned} \text{total time} &= \sum_{u, v \in V} H(u, v) + H(v, u) \\ &= 2m \sum_{u, v \in V} (\text{energy of unit flow } (u, v)) \leq 2m(n-1) \end{aligned}$$



electrical energy = $\sum_{e \in E} r_e f_e^2$
electrical flow = least energy

des $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
(BS) $_v$ = net flow at v
 $d_v = \begin{cases} -1 & \text{if } v = s \\ 1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}$

min $\frac{1}{2} \sum_{e \in E} r_e f_e^2$ s.t. $BS = d$
(electrical energy) (is a typical)

effective resistance of d = optimum value/energy

Ohm's law: $f = R^{-1} B^T p$
for electric flow by electric potentials p

$L = BR^{-1}B^T \in \mathbb{R}^{V \times V}$
(effective resist. of d) $= \langle d, L^{-1}d \rangle$

Lecture 19

Probability (and techniques)

Algorithms (models and applications)

Spectral theorem
for symmetric maps

Spectral analysis of
random walks

Lecture 20

Probability
(and techniques)

Algorithms
(models and applications)

Conductance

Cheeger's inequality

Conductance-based
mixing time

Fiedler's algorithm
for low-conductance
cuts

Lecture 21

Probability
(and techniques)

Algorithms
(models and applications)

Expanders

Zigzag product

tensors

Deterministic

Connectivity in

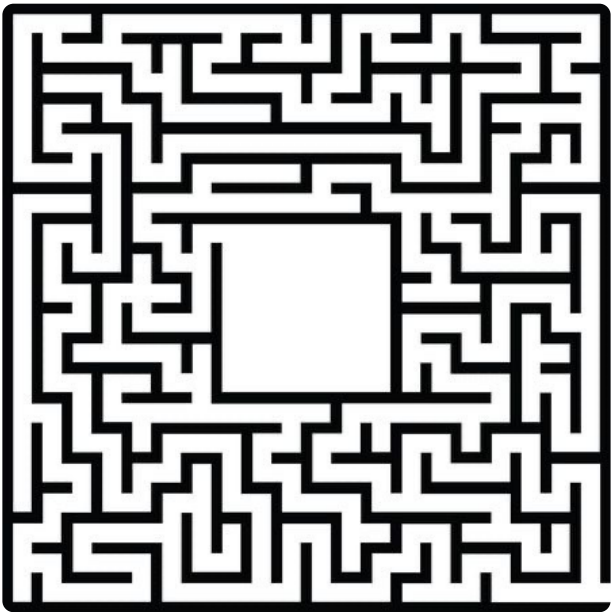
$O(\log n)$ space

connectivity

Input: $G=(V,E)$ undirected,
(m edges, n vertices)

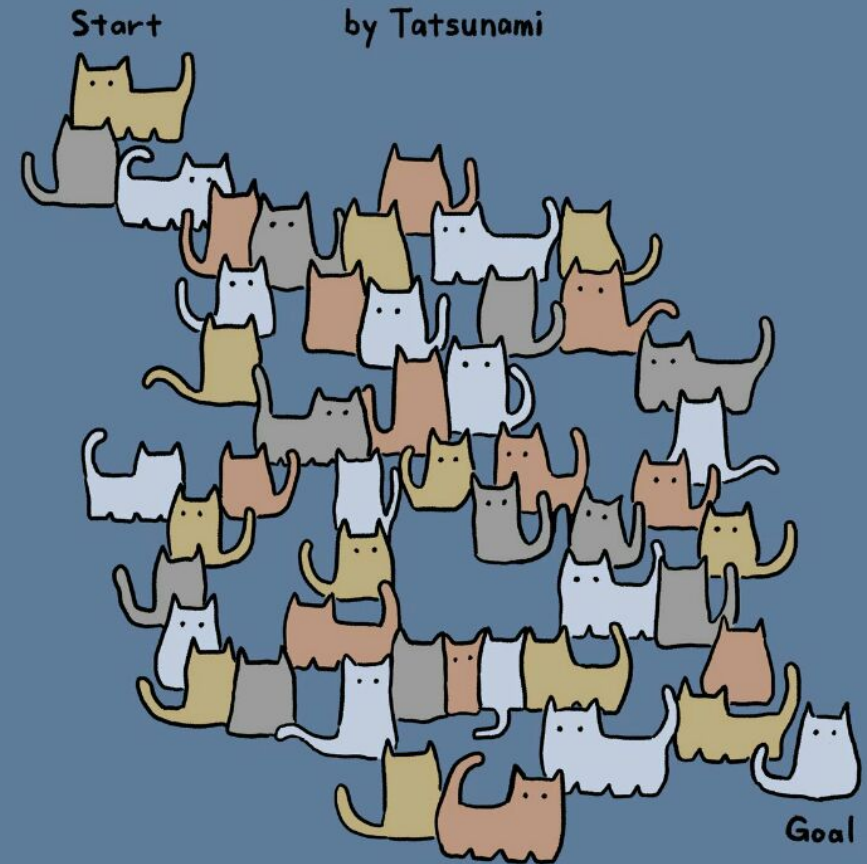
$s, t \in V$

Q: is there a path from
s to t?



CAT MAZE

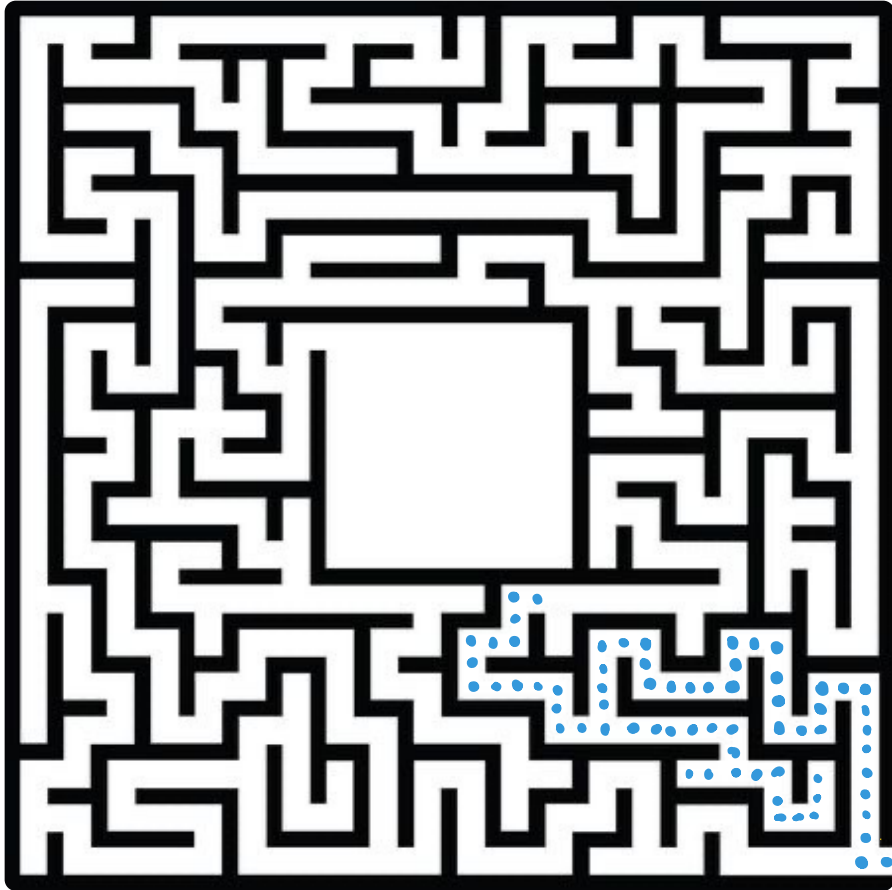
by Tatsunami



Reach the Goal by following the cats, alternating between cats facing the front and cats facing backwards.

The cats in your path must be connected or overlapping.

It doesn't matter what color or size they are.



Breadcrumbs

BFS, DFS take
linear time and space

can connectivity be solved in...

Lies

Yes

No

Not
voting

linear space?

BFS,
DFS

$O(\log^{O(1)} n)$ space?

90%

1

3

$O(\log n)$ space?

90%

4

2

$O(\log n)$ space
deterministically?

50%

50%

2 weeks
from now

can connectivity be solved in $O(\log n)$ space?

algorithm:

- (1) repeat $O(\log n)$ times:
 - (a) start at $s \in V$
 - (b) randomly walk for $O(mn)$ steps.
 - (c) if walk visits t , return TRUE
- (2) return False

random walk terminology:

Hitting time $H(s, t)$ = expected # steps until a random walk from s reaches t

Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex

Theorem

If G is connected, then $C(s) \leq m(n-1)$

Q: how about w/out randomization?

Recall

$$\text{score}_3(v) = \sum_{(u,v) \in E} \frac{\text{score}_3(u)}{\deg^+(u)}$$



let $\Delta^V = \{x \in \mathbb{R}_V^{\geq 0} : \sum_v x_v = 1\} = \{\text{distributions over } V\}$

Theorem there exists $x \in \Delta^V$ s.t.

$$x_v = \sum_{(u,v) \in E} \frac{x_u}{\deg^+(u)} \quad \forall v \in V.$$

If G is strongly connected, then x is unique and strictly positive.

Theorem, restated: every random walk has a stationary distribution. If G is strongly connected, then this distribution is unique.

Theorem, restated: let G be a dir. graph and R the random walk linear map. Then R has an eigenvector $x \in \Delta^V$ w/ eigenvalue 1. If G is strongly connected, then x is the unique eigenvector w/ eigenvalue 1

Recall

Theorem: $G=(V,E)$ undirected, $w:E \rightarrow \mathbb{R}_{>0}$
 let $R:\mathbb{R}^V \rightarrow \mathbb{R}^V$ be random walk map
 then R has stationary distr. $s \in \Delta_V$ w/

$$s_v = \frac{\deg(v)}{2W}$$

$$\deg(v) = \sum_{e \in \partial(v)} w(e)$$

$$W = \sum_{e \in E} w(e)$$

Proof $d \in \mathbb{R}^V$ be the degrees

$$D = \text{diag}(d)$$

$$A = \text{adj. matrix} \in \mathbb{R}^{n \times n}$$

$$R = AD^{-1}$$

$$Rd = AD^{-1}d = A\mathbf{1} = d$$

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) \rangle = \langle x, u \rangle \langle v, x \rangle$

Laplacian of $e = \{u, v\}$
 $L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{V \times V}$
($1_u = \{0,1\}$ -indicator vector for u)
 $\langle x, L_e x \rangle = (x_u - x_v)^2$

Laplacian of 
 $L = \sum_{e \in E} w(e) L_e$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

positive semi-definite A :
 • A is symmetric
 • $\langle x, Ax \rangle \geq 0 \quad \forall x$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ maximizes $\langle x, Lx \rangle$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$
 where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

Recall

$$Q = \frac{\sqrt{d} \otimes \sqrt{d}}{\langle \mathbf{1}, d \rangle} + \mu_2 (v_2 \otimes v_2) + \dots + \mu_n (v_n \otimes v_n)$$

w/ $\mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n$
(also eigenvalues of R)

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary distribution

Rank-1 matrix
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 $v_1, \dots, v_n$ orthonormal basis

Theorem: $G=(V, E)$ undirected, $w: E \rightarrow \mathbb{R}_{>0}$
then R has stationary distr. $s \in \Delta_V$ w/

$$s_v = \frac{\deg(v)}{2W} \quad \deg(v) = \sum_{e \in \partial(v)} w(e), \quad W = \sum_{e \in E} w(e)$$

Lemma let $A = C^{-1}BC$.

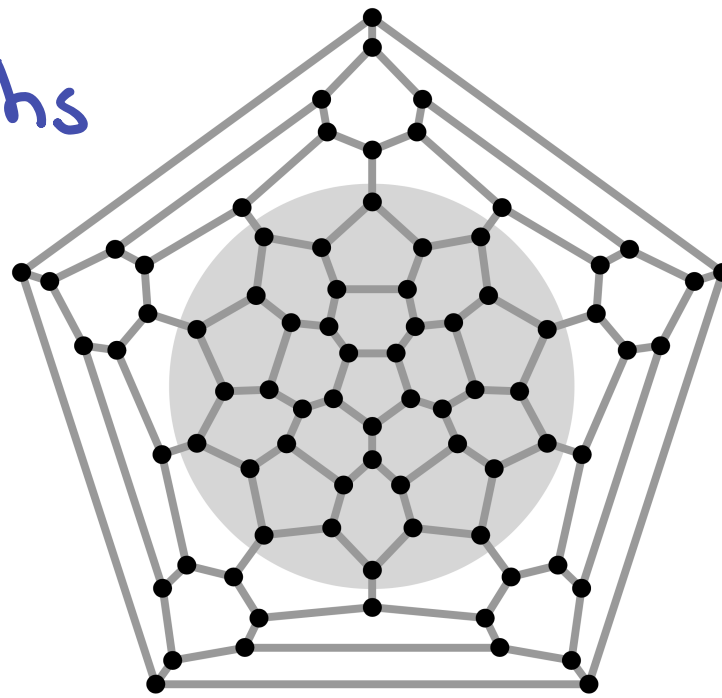
- A and B have same eigenvalues
- C maps eigenvectors of A to eigenvectors of B

Normalized random walk matrix

$$Q = D^{-1/2} R D^{1/2} = D^{-1/2} A D^{-1/2}$$

- $D = \text{diag}(d)$ w/ $d = \text{degrees}$
- $A = \text{adjacency matrix}$
- $R = A D^{-1} = \text{random walk matrix}$
- Q is symmetric!
- Q has same eigenvalues as R !

Expander graphs



Theorem, restated: let G be a dir. graph and R the random walk linear map. Then R has an eigenvector $x \in \Delta^V$ w/ eigenvalue 1. If G is strongly connected, then x is the unique eigenvector w/ eigenvalue 1

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↑
stationary
distribution

$1 \geq \mu_2 \geq \dots \geq \mu_n = \text{eigenvalues of } R$

G is an expander if the spectral gap δ is at least a constant

Q: deterministic connectivity for constant-degree expanders?

Q: diameter of a constant-degree expander?

Q: diameter of a
constant-degree expander?

say $\gamma = 1/2$. let $x = 1_s$ for $s \in V$

$$\|R^k x - \underset{\substack{\uparrow \\ \text{stationary} \\ \text{distribution}}}{\frac{d}{2m}}}\|_2 \leq \left(\frac{1}{2}\right)^k \cdot n$$

for $k = O(\log n)$,

$$\|R^k x - \frac{d}{2m}\|_\infty \leq \|R^k x - \frac{d}{2m}\|_2 \leq \frac{1}{\text{poly}(n)}$$

$$\Rightarrow (R^k x)_v \geq \frac{d}{2m} - \frac{1}{\text{poly}(n)} > 0 \quad \forall v$$

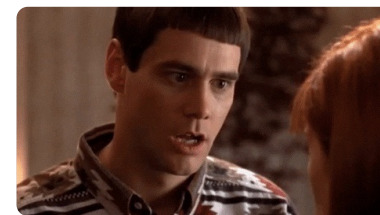
$\Rightarrow$ there exists a path of $O(\log n)$ edges from s to t

Theorem, restated: let G be a dir. graph and R the random walk linear map. Then R has an eigenvector $x \in \Delta^V$ w/ eigenvalue 1. If G is strongly connected, then x is the unique eigenvector w/ eigenvalue 1

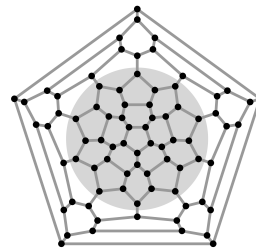
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 $W = \sum_{e \in E} w(e)$

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$
 $\|R^k x - s\| \leq (1-\gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$
 $\uparrow$
stationary distribution
 $1 \geq \mu_2 \geq \dots \geq \mu_n = \text{eigenvalues of } R$

G is an expander if the spectral gap γ is at least a constant



Expander graphs



G is an expander if the spectral gap γ is at least a constant

Theorem, restated: let G be a dir. graph and R the random walk linear map. Then R has an eigenvector $x \in \Delta^V$ w/ eigenvalue 1. If G is strongly connected, then x is the unique eigenvector w/ eigenvalue 1

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 s stationary distribution
 $1 \geq \mu_2 \geq \dots \geq \mu_n = \text{eigenvalues of } R$

Q: deterministic connectivity for constant-degree expanders?

✓ Q: diameter of a constant-degree expander?
 $O(\log n)$

Q: how many random bits are needed for 1 step of a random walk in a constant-degree graph?

Expander graphs

G is an expander if the spectral gap γ is at least a constant

Theorem, restated: let G be a dir. graph and R the random walk linear map. Then R has an eigenvector $x \in \Delta^V$ w/ eigenvalue 1. If G is strongly connected, then x is the unique eigenvector w/ eigenvalue 1

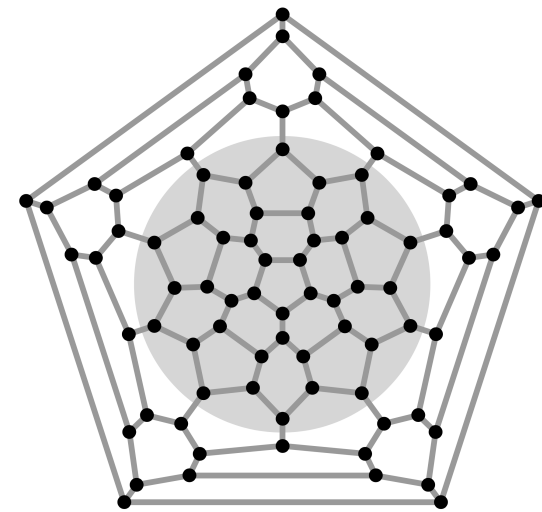
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Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$
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stationary distribution
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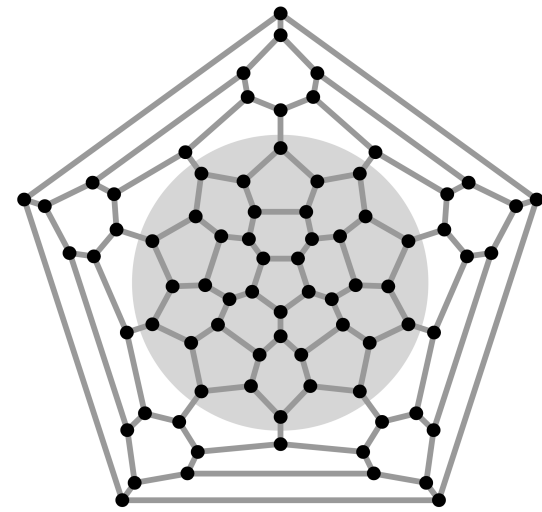
Q: deterministic connectivity for constant-degree expanders?

- Diameter $O(\log n)$
- $O(1)$ random bits per random step

$\Rightarrow$ enumerate all paths of length $O(\log n)$!



Lemma: $O(\log n)$ space, poly-time
deterministic connectivity in
 $O(1)$ -degree expanders



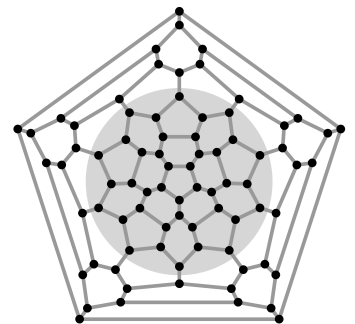
[Reingold, 2005]

Theorem: $O(\log n)$ space, poly-time
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Theorem: $O(\log n)$ space, poly-time
deterministic connectivity in undirected graphs. [Reingold, 2005]

High level idea:

- convert G into $O(1)$ -degree expander
by sequence of "graph transformations"
(graph transformations applied implicitly due to space constraints)
- derandomize the random walk
algo (implicitly) on the expander

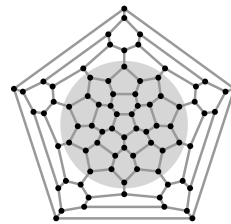


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High level idea:

- Convert G into $O(1)$ -degree expander by sequence of "graph transformations"
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- derandomize the random walk algo (implicitly) on the expander



Lemma: $O(\log n)$ space, poly-time
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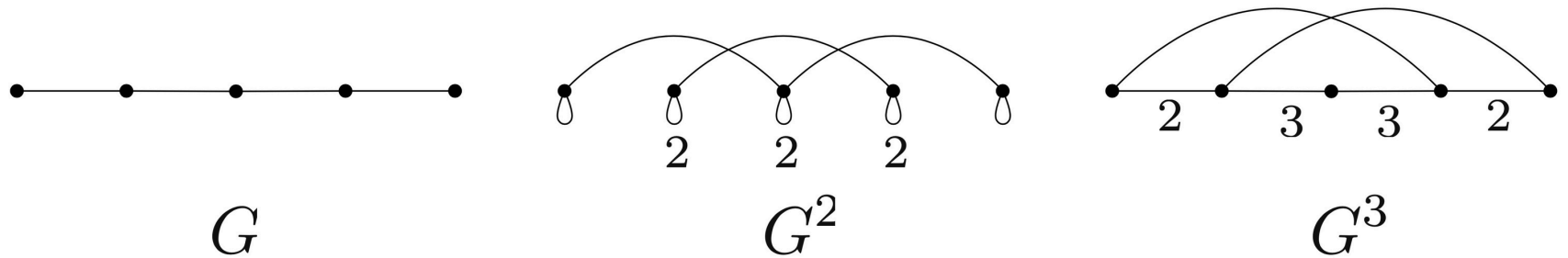
Graph transformations:

1. Powering
2. Zig-Zag product

Preliminary

WLOG assume G is a regular graph
of degree $d = O(1)$

Powering



$G^k =$ graph w/ an edge for every k step walk in G
(w/ repetition)

random walk R^k has eigenvalues $1, u_2^k, \dots, u_n^k$

$$\delta(G^k) = 1 - (1 - \delta(G))^k \quad \text{😊}$$

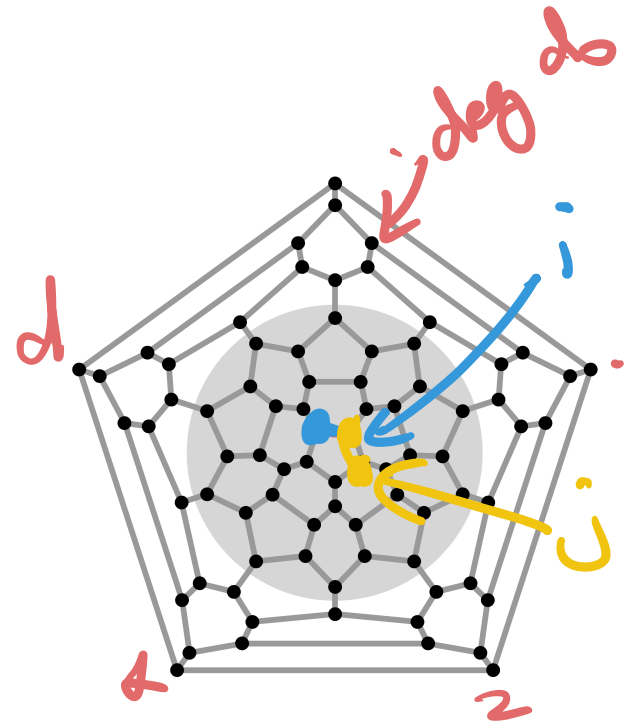
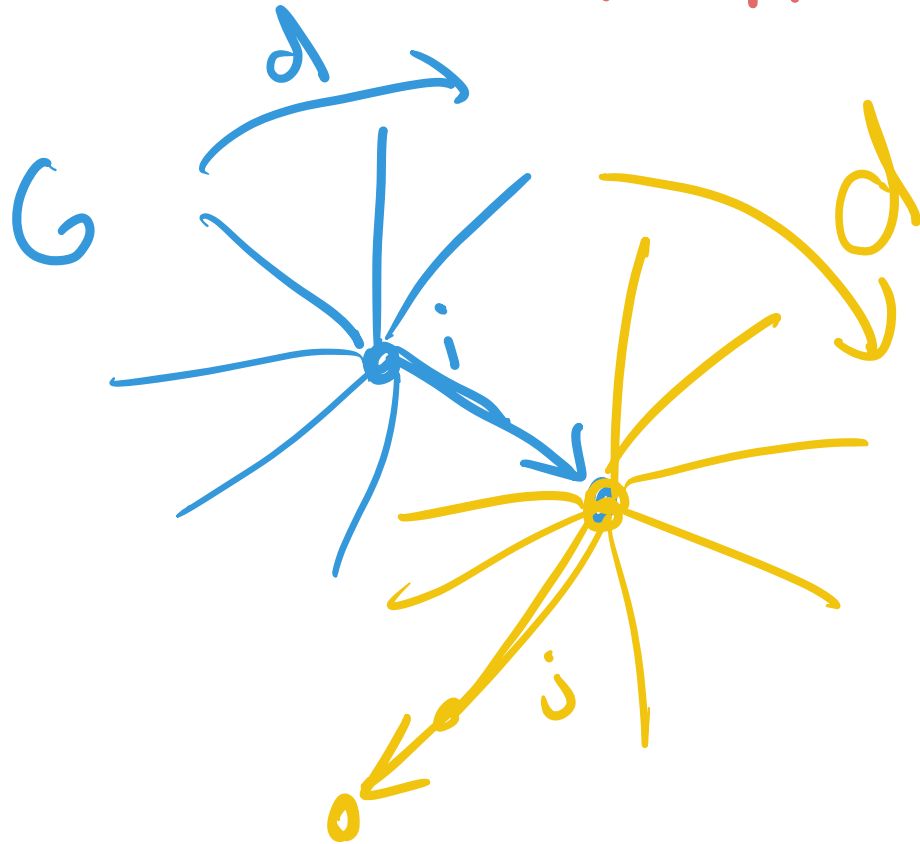
degree increases from d to d^k 😞

Zig-zag product $Z(G, H)$

let G be d -regular

Problem: d large (after powering)

how to simulate "random step" w/ fewer bits?



Zig-zag product $Z(G, H)$

let G be d -regular

H be d_0 -regular w/ d_0 vertices (H will be $O(D)$ -degree expander)

vertices: $V_G \times V_H = \{(u, v) : u \in V_G, v \in V_H\}$

edges: for $(v_1, i_1) \in V_G \times V_H$, $k_1, k_2 \in [d_0]$:

the (k_1, k_2) th neighbor of (v_1, i_1) is (v_2, i_4) where:

$$\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} \xrightarrow{(a)} \begin{pmatrix} v_1 \\ i_2 \end{pmatrix} \xrightarrow{(b)} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix} \xrightarrow{(c)} \begin{pmatrix} v_2 \\ i_3 \end{pmatrix} \xrightarrow{(d)} \begin{pmatrix} v_2 \\ i_4 \end{pmatrix}$$

where:

- (a) is a step in H from i_1 to its k_1 th neighbor, i_2 .
- (b) moves v_1 to the i_2 th outgoing neighbor of v_1 , v_2 .
- (c) moves from i_2 to i_3 where v_1 is the i_3 th neighbor of v_2 .
- (d) moves i_3 to its k_2 th neighbor in H .

(so $Z(G, H)$ is d_0^2 -regular)

Lemma: $Z(G|H)$ has

- n vertices
- degree d_0^2 😊
- spectral gap $\gamma_G \gamma_H^2$ 😊

Zig-zag product $Z(G,H)$

let G be d -regular

H be d_0 -regular w/ d vertices (H will be $O(1)$ -degree expander)

vertices: $V_G \times V_H = \{(u,v) : u \in V_G, v \in V_H\}$

edges: for $(v_1, i_1) \in V_G \times V_H$, $k_1, k_2 \in [d_0]$:

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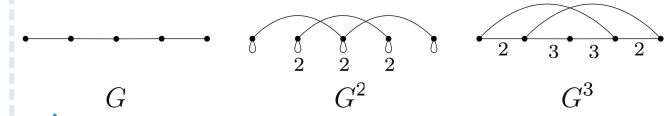
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(so $Z(G,H)$ is d_0^2 -regular)

	G^k	$Z(G, H)$
vertices	n	nd
degree	d^k	d_0^2
spectral gap	$1 - (1 - \gamma_G)^k$	$\gamma_G \gamma_H^2$

	$Z(G^2, H)$
vertices	nd^2
degree	d_0^2
spectral gap	$(1 - (1 - \gamma_G^2)) \gamma_H^2 = (2\gamma_G - \gamma_G^2) \cdot \gamma_H^2$ $= 2\gamma_G \gamma_H^2 - \gamma_G^2 \gamma_H^2$

Powering



G^k = graph w/ edge for every k step walk in G
(w/ repetition)

random walk R^k has eigenvalues $1, \mu_1^k, \dots, \mu_n^k$

$$\gamma(G^k) = 1 - (1 - \gamma(G))^k \quad \text{😊}$$

degree increases from d to d^k 😞

Zig-zag product $Z(G, H)$

let G be d -regular

H be d_0 -regular w/ d vertices (H will be $O(d)$ -degree expander)

vertices: $V_G \times V_H = \{(u, v) : u \in V_G, v \in V_H\}$

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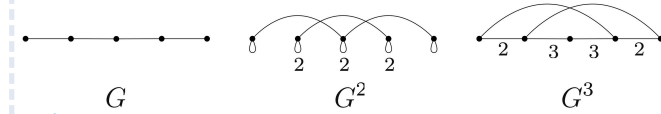
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	G^k	$Z(G, H)$
vertices	n	nd
degree	d^k	d_0^2
spectral gap	$1 - (1 - \gamma_G)^k$	$\gamma_G \gamma_H^2$
	$Z(G^2, H)$	
vertices	nd	
degree	d_0^2	
spectral gap	$(1 - (1 - \gamma_G^2)) \gamma_H^2$	

Powering



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degree increases from d to d^k 😞

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where:

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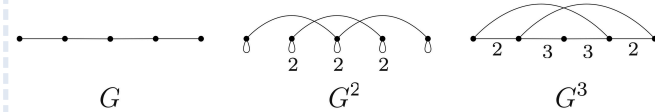
(so $Z(G, H)$ is d_0^2 -regular)

	$Z(G^2, H)$
vertices	nd
degree	d_0^2
spectral gap	$(1 - (1 - \gamma_G^2)) \gamma_H^2$

for $\gamma_G \leq \frac{1}{16}$, $\gamma_H \geq 3/4$,

$$(1 - (1 - \gamma_G^2)) \gamma_H^2 \geq 1.01 \gamma_G$$

Powering



G^k = graph w/ edge for every k step walk in G
(w/ repetition)

random walk R^k has eigenvalues $1, \mu_1^k, \dots, \mu_n^k$

$$\gamma(G^k) = 1 - (1 - \gamma(G))^k \quad \text{😊}$$

degree increases from d to d^k 😞

Zig-zag product $Z(G, H)$

let G be d -regular

H be d_0 -regular w/ d vertices (H will be $O(d)$ -degree expander)

vertices: $V_G \times V_H = \{(u, v) : u \in V_G, v \in V_H\}$

edges: for $(v_1, i_1) \in V_G \times V_H$, $k_1, k_2 \in [d_0]$:

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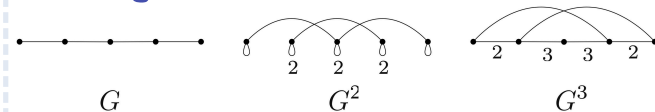
	G^k	$Z(G, H)$
vertices	n	nd
degree	d^k	d_0^2
spectral gap	$1 - (1 - \gamma_G^k)^k$	$\gamma_G \gamma_H^2$

Space usage

for graphs like G^k or $Z(G, H)$,
need to answer in $O(1)$ space:

given v , what is i th neighbor
of v

Powering



G^k = graph w/ edge for every k step walk in G
(w/ repetition)

random walk R^k has eigenvalues $1, \mu_1^k, \dots, \mu_n^k$

$$\chi(G^k) = 1 - (1 - \chi(G))^k \quad \text{😊}$$

degree increases from d to d^k 😞

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H be d_0 -regular w/ d vertices (H will be $O(d)$ -degree expander)

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vertices	n	nd
degree	d^k	d_0^2
spectral gap	$1 - (1 - \chi_G)^k$	$\chi_G \chi_H^2$

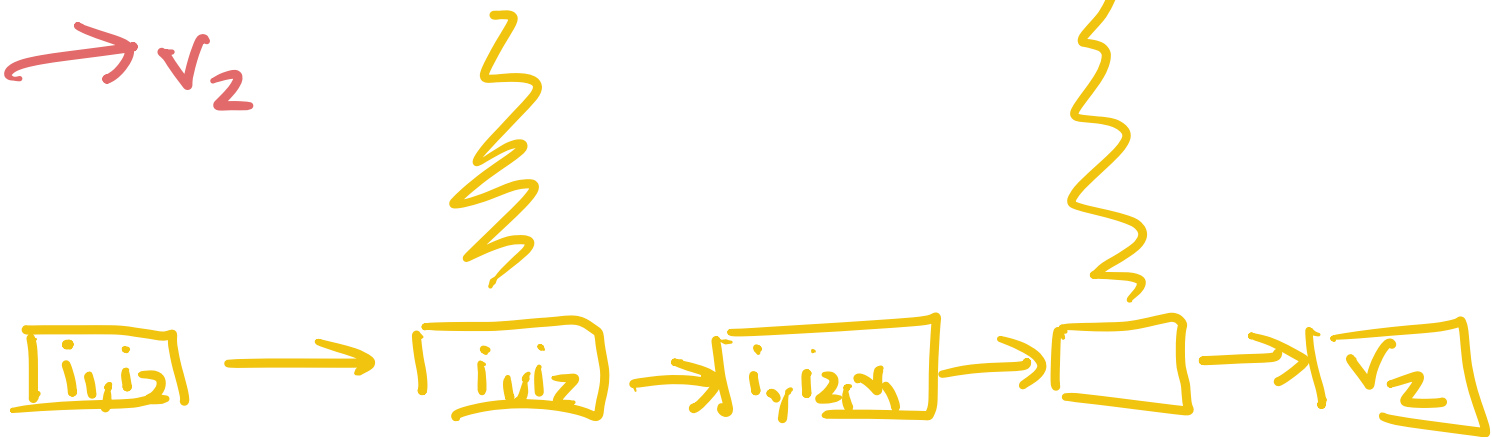
	$Z(G^2, H)$
vertices	nd
degree	d_0^2
spectral gap	$(1 - (1 - \chi_G^2)) \chi_H^2$

(i_1, i_2) th neighbor of v in G^2 ?
stack



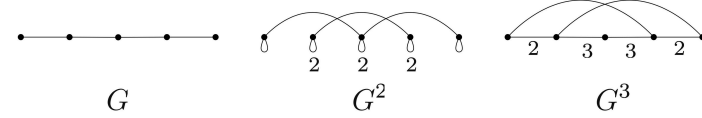
1. i_1 th neighbor of v in G
 $\rightarrow v_1$ rec

2. i_2 th neighbor of v_1 in G
 $\rightarrow v_2$



Zig zag

Powering



G^k = graph w/ an edge for every k step walk in G
(w/ repetition)

random walk R^k has eigenvalues $1, u_1^k, \dots, u_n^k$

$$\chi(G^k) = 1 - (1 - \chi(G))^k \quad \text{😊}$$

degree increases from d to d^k 😞

(k_1, k_2) th neighbor of (v_1, i_1) ?

1 k_2 th neighbor of i_1
↓
 i_2

2 i_2 th neighbor of v_1 in G
rec. call to neighbor in G
↓
 v_2

3 loop: j th neighbor of v_2 in G until v_1

4. k_2 nd neighbor of v_1

Zig-zag product $Z(G, H)$

let G be d -regular

(H will be $O(1)$ -degree expander)

H be d_0 -regular w/ d vertices

vertices: $V_G \times V_H = \{(u, v) : u \in V_G, v \in V_H\}$

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(so $Z(G, H)$ is d_0^2 -regular)

Analysis of zig-zag product

let R_H = random walk for H

Zig-zag product $Z(G, H)$

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(so $Z(G, H)$ is d^2 -regular)

write random walk for $Z(G, H)$ as

$$(I \otimes R_H) Z (I \otimes R_H)$$

where:

- $(I \otimes R_H)$ takes random step in H ,
leaves G -coordinate fixed
- Z does deterministic "zig-zag" step

Simpler zig-zag product

$$\text{let } S = (I \otimes 1)/d$$

= uniformly sample 1 vertex in H

consider

$$(I \otimes S)Z(I \otimes S) = R_G \otimes S$$

where:

- $(I \otimes S)$ randomly samples vertex in H
- Z uses H -coordinate to take step in G

Zig-zag product $Z(G, H)$

let G be d -regular

(H will be $O(d)$ -degree expander)
 H be d_0 -regular w/ d vertices

vertices: $V_G \times V_H = \{(u, v) : u \in V_G, v \in V_H\}$

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Analysis of zig-zag product

let R_H = random walk for H

write random walk for $Z(G, H)$ as

$$(I \otimes R_H) Z (I \otimes R_H)$$

where:

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- Z uses H -coordinate to take step in G

Simpler zig-zag product

$$\text{let } S = (1 \otimes 1)/d$$

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consider

$$(I \otimes S) Z (I \otimes S)$$

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- $(I \otimes S)$ samples vertex in H

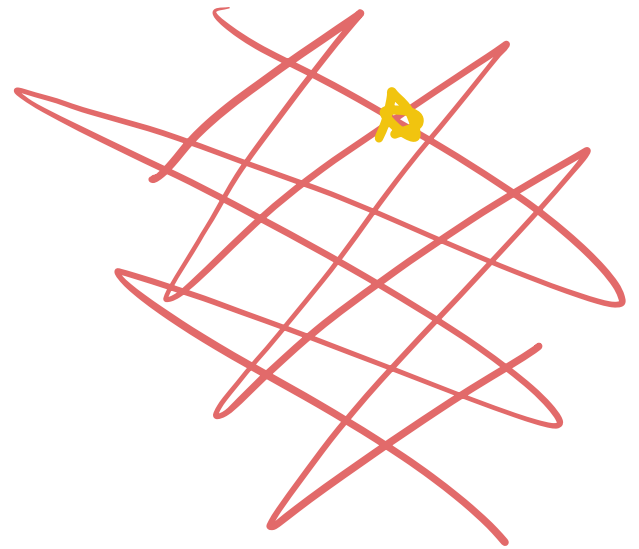
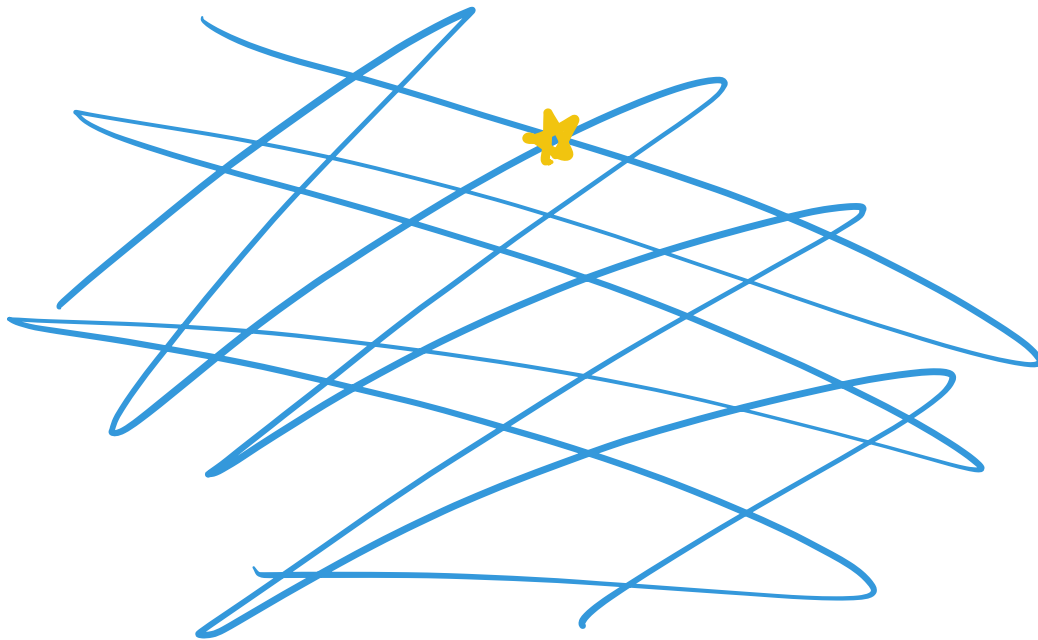
Idea: if H expander, then random walk on H behaves similarly to random sample.

$$\Rightarrow (I \otimes R_H) Z (I \otimes R_H) \approx \underbrace{(I \otimes S) Z (I \otimes S)}_{\text{good spectral gap}} = R_G \otimes I_S$$

Tensor products of graphs

$$G_1 \otimes G_2 = (V_1 \times V_2, E_1 \times E_2)$$

$$E_1 \times E_2 = \{((u_1, u_2), (v_1, v_2)) : (u_1, v_1) \in E_1, (u_2, v_2) \in E_2\}$$



Tensor of vectors: Rank-1 matrix

$$u \otimes v \quad \text{where } u, v \in \mathbb{R}^n$$

(aka. uv^T)

defined by $(u \otimes v)y = \langle v, y \rangle u$

as a matrix:

$$u \otimes v = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_n v_1 & u_n v_2 & & u_n v_n \end{bmatrix}$$

$$\otimes \text{ maps } u, v \in \mathbb{R}^n \rightarrow u \otimes v \in \mathbb{R}^{n \times n} = \mathbb{R}^n \rightarrow \mathbb{R}^n$$

Tensor products

for $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{n \times n}$,

$A \otimes B$ is a linear map on $\mathbb{R}^{m \times n}$ defined by

$$(A \otimes B)C = \tilde{A} C B^T \quad \text{for } C \in \mathbb{R}^{m \times n}$$

special case:

$$(A \otimes B)(c \otimes d) = (Ac) \otimes (Bd) \quad c \in \mathbb{R}^m, d \in \mathbb{R}^n$$

Fact: $A \otimes B$ has eigenvalues $\lambda_A \lambda_B$ where

λ_A is an eigenvalue of A ,

λ_B B

Tensor product $I \otimes R_H$

$I \otimes R_H$ is a linear map on $\mathbb{R}^{V \times d}$ defined by

$$((I \otimes R_H)X)_v = R_H X_v \quad X \in \mathbb{R}^{V \times d}$$

where $X_v = v$ th row of X .

Fact: $I \otimes (A+B) = I \otimes A + I \otimes B$

$$R_H = v_1 \otimes v_1 + \mu_2(v_2 \otimes v_2) + \dots + \mu_n(v_n \otimes v_n)$$

$$R_H \mathbf{1} = \mathbf{1} \Rightarrow v_i = \mathbf{1}/\sqrt{d},$$

$$v_i \otimes v_i = (\mathbf{1} \otimes \mathbf{1})/d = S$$

$$R_H = S + \mu_2(v_2 \otimes v_2) + \dots + \mu_n(v_n \otimes v_n)$$

$$\text{let } R'_H = R_H - \gamma_H S \quad \gamma_H = \max \{1 - \mu_2, 1 - \mu_n\}$$

R'_H has eigenvalues in $[\gamma_H - 1, 1 - \gamma_H]$

Zig-zag product $Z(G, H)$

let G be d -regular (H will be $O(d)$ -degree expander)
 H be d_0 -regular w/ d vertices

vertices: $V_G \times V_H = \{(u, v) : u \in V_G, v \in V_H\}$

edges: for $(v_1, i_1) \in V_G \times V_E$, $k_1, k_2 \in [d_0]$:
 the (k_1, k_2) th neighbor of (v_1, i_1) is (v_2, i_4)

$$\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} \xrightarrow{(a)} \begin{pmatrix} v_1 \\ i_2 \end{pmatrix} \xrightarrow{(b)} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix} \xrightarrow{(c)} \begin{pmatrix} v_2 \\ i_3 \end{pmatrix} \xrightarrow{(d)} \begin{pmatrix} v_2 \\ i_4 \end{pmatrix}$$

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(so $Z(G, H)$ is d^2 -regular)

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$$(I \otimes R_H) Z (I \otimes R_H)$$

$$= (I \otimes (R'_H + \gamma_H S)) Z (I \otimes (R'_H + \gamma_H S))$$

$$= (I \otimes R'_H) Z (I \otimes R'_H)$$

$$+ \gamma_H (I \otimes S) Z (I \otimes R'_H)$$

$$+ \gamma_H$$

$$+ \gamma_H^2$$



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$$(I \otimes R_H) Z (I \otimes R_H)$$

$$= \gamma^2 (I \otimes S) Z (I \otimes S)$$

$$+ \gamma_H (I \otimes S) Z (I \otimes R'_H)$$

$$+ \gamma_H (I \otimes R'_H) Z (I \otimes S)$$

$$+ (I \otimes R'_H) Z (I \otimes R'_H)$$

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$$I \otimes (A + B) = I \otimes A + I \otimes B$$

Operator norm:

$$\|A\| = \max \{ \|Ax\| : \|x\| = 1 \}$$

helpful inequalities:

$$\|A+B\| \leq \|A\| + \|B\|$$

$$\|AB\| \leq \|A\| \|B\|$$

$$\|A \otimes B\| \leq \|A\| \|B\|$$

for symmetric A, B

Fact: $A \otimes B$ has eigenvalues $\lambda_A \lambda_B$ where
 λ_A is an eigenvalue of A ,
 λ_B is an eigenvalue of B

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$$+ \gamma_H (I \otimes S) Z (I \otimes R'_H)$$

$$+ \gamma_H (I \otimes R'_H) Z (I \otimes S)$$

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$$I \otimes (A + B) = I \otimes A + I \otimes B$$

$$\|A + B\| \leq \|A\| + \|B\|$$

$$\|AB\| \leq \|A\| \|B\|$$

$$\|A \otimes B\| \leq \|A\| \|B\|$$

$R'_H = R_H - \gamma_H S$ has eigenvalues in $[\gamma_H - 1, 1 - \gamma_H]$

$$\begin{aligned} & (I \otimes R_H) Z (I \otimes R_H) \\ &= \gamma_H^2 (I \otimes S) Z (I \otimes S) \\ &+ \gamma_H (I \otimes S) Z (I \otimes R'_H) \\ &+ \gamma_H (I \otimes R'_H) Z (I \otimes S) \\ &+ (I \otimes R'_H) Z (I \otimes R'_H) \end{aligned}$$

random vertex in H

$$(I \otimes S) Z (I \otimes S)$$

step in G based on H

$\Rightarrow$ $\left. \begin{array}{l} \bullet \text{ random step in } G \\ \bullet \text{ random vertex in } H \end{array} \right\} R_G \otimes S$

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$$(I \otimes S) Z (I \otimes S) = R \otimes S$$

Fact: $A \otimes B$ has eigenvalues $\lambda_A \lambda_B$ where

λ_A is an eigenvalue of A ,

λ_B is an eigenvalue of B

Zig-zag product $Z(G, H)$

let G be d -regular

(H will be $O(d)$ -degree expander)

H be d_0 -regular w/ d vertices

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$$(I \otimes S) Z (I \otimes R'_H)$$

max

$$\|(I \otimes S) Z (I \otimes R'_H) x\| \leq 1 - \gamma_H$$

s.t. ~~x orthogonal $\mathbb{1} \otimes \mathbb{1}$~~

work in space
orthogonal
 ~~$\mathbb{1}, \mathbb{1}$~~

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 \end{aligned}$$

$$\underbrace{\| (I \otimes R'_H) Z (I \otimes R'_H) \|}_{\mu_2} \leq \mu_2^2 = (1 - \gamma_H)^2$$

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All put together:

For $x \in \mathbb{R}^{v \times d}$ orthogonal to $\mathbb{1}^{v \times d}$, $\|x\|=1$,

$$\langle x, (I \otimes R_H) Z (I \otimes R_H) x \rangle$$

$$= \gamma_H^2 \langle x, (I \otimes S) Z (I \otimes S) x \rangle \leq \langle x, (R_G \otimes S) x \rangle \leq (1 - \gamma_G)$$

$$+ \gamma_H \langle x, (I \otimes S) Z (I \otimes R'_H) x \rangle \leq \cancel{\gamma_H} (1 - \gamma_H)$$

$$+ \gamma_H \langle x, (I \otimes R'_H) Z (I \otimes S) x \rangle \leq 1 - \gamma_H$$

$$+ \langle x, (I \otimes R'_H) Z (I \otimes R'_H) x \rangle \leq (1 - \gamma_H^2)^2$$

$$\boxed{\gamma_H^2} (1 - \gamma_G) + 2\gamma_H \cdot (1 - \gamma_H) + (1 - \gamma_H)^2$$

$\downarrow$ $\downarrow$ $\downarrow$
 $1 - \gamma_G \gamma_H^2$ $\boxed{-2\gamma_H^2}$ $-2\gamma_H$ γ_H^2